

On condensation properties of Bethe roots associated with the XXZ chain

Karol K. Kozłowski¹

IMB, UMR 5584 du CNRS, Université de Bourgogne, France.

Abstract

I prove that the Bethe roots describing either the ground state or a certain class of "particle-hole" excited states of the XXZ spin-1/2 chain in any sector with magnetisation $m \in [0; 1/2]$ exist and form, in the infinite volume limit, a dense distribution on a subinterval of \mathbb{R} . The results holds for any value of the anisotropy $\Delta \geq -1$. In fact, I establish an even stronger result, namely the existence of an all order asymptotic expansion of the counting function associated with such roots. As a corollary, these results allow one to prove the existence and form of the infinite volume limit of various observables attached to the model -the excitation energy, momentum, the zero temperature correlation functions, so as to name a few- that were argued earlier in the literature.

Introduction

The XXZ spin-1/2 chain refers to a system of interacting spins in one dimension described by the Hamiltonian

$$H_\Delta = J \sum_{a=1}^L \left\{ \sigma_a^x \sigma_{a+1}^x + \sigma_a^y \sigma_{a+1}^y + \Delta \sigma_a^z \sigma_{a+1}^z \right\}. \quad (0.1)$$

H_Δ is an operator on the Hilbert space of the model $\mathfrak{h}_{XXZ} = \otimes_{a=1}^L \mathfrak{h}_a$ with $\mathfrak{h}_a \simeq \mathbb{C}^2$. The matrices σ^w , $w = x, y, z$ are the Pauli matrices and σ_a^w stands for the operator on \mathfrak{h}_{XXZ} which acts as the Pauli matrix σ^w on \mathfrak{h}_a and as the identity on all other spaces appearing in the tensor product defining \mathfrak{h}_{XXZ} . The Hamiltonian depends on two coupling constants : $J > 0$ which represents the so-called exchange interaction and Δ which takes into account the anisotropy in the coupling between the spins in the longitudinal and transverse directions. Finally, the chain consists of an even number of sites $L \in 2\mathbb{N}$.

The Hamiltonian H_Δ commutes with the total spin operator

$$S^z = \sum_{a=1}^L \sigma_a^z. \quad (0.2)$$

1. karol.kozlowski@u-bourgogne.fr

The Hilbert space of the model \mathfrak{h}_{XXZ} decomposes into the direct sum

$$\mathfrak{h}_{XXZ} = \bigoplus_{N=0}^L \mathfrak{h}_{XXZ}^{(N)} \quad \text{with} \quad \mathfrak{h}_{XXZ}^{(N)} = \left\{ \mathbf{v} \in \mathfrak{h}_{XXZ} : S^z \cdot \mathbf{v} = (L - 2N) \cdot \mathbf{v} \right\}, \quad (0.3)$$

what turns H_Δ into a block diagonal operator relatively to the above decomposition. I denote by $H_\Delta^{(N)}$ the restriction of H_Δ to every subspace $\mathfrak{h}_{XXZ}^{(N)}$.

The spectrum and eigenvectors of the isotropic limit $\Delta = 1$ of the XXZ chain have been first studied by Bethe [2] in 1931 by means of the celebrated Bethe Ansatz. Then, in 1958, Orbach extended the approach to the case of the XXZ chain [30] with a general coupling Δ . Within Bethe's Ansatz, the eigenvectors and eigenvalues of $H_\Delta^{(N)}$ are parametrised by a collection of N numbers $\{\lambda_a\}_1^N$, the so-called Bethe roots. The Bethe roots are constrained to satisfy a system of transcendental equations called the Bethe equations. For $-1 < \Delta < 1$ when the anisotropy can be parametrised as $\Delta = \cos(\zeta)$ with $\zeta \in]0; \pi[$, these take the form

$$\prod_{b=1}^N \left\{ \frac{\sinh(\lambda_a - \lambda_b + i\zeta)}{\sinh(\lambda_b - \lambda_a + i\zeta)} \right\} \cdot \left(\frac{\sinh(i\zeta/2 - \lambda_a)}{\sinh(i\zeta/2 + \lambda_a)} \right)^L = (-1)^{N+1}, \quad a = 1, \dots, N. \quad (0.4)$$

When $\Delta = 1$, the Bethe equations degenerate into the rational form

$$\prod_{b=1}^N \left\{ \frac{\lambda_a - \lambda_b + i}{\lambda_b - \lambda_a + i} \right\} \cdot \left(\frac{i/2 - \lambda_a}{i/2 + \lambda_a} \right)^L = (-1)^{N+1}, \quad a = 1, \dots, N. \quad (0.5)$$

Finally, for $\Delta > 1$, the anisotropy can be parametrised as $\Delta = \cosh(\zeta)$ with $\zeta \in \mathbb{R}^+$ and the Bethe equations take the form

$$\prod_{b=1}^N \left\{ \frac{\sin(\lambda_a - \lambda_b + i\zeta)}{\sin(\lambda_b - \lambda_a + i\zeta)} \right\} \cdot \left(\frac{\sin(i\zeta/2 - \lambda_a)}{\sin(i\zeta/2 + \lambda_a)} \right)^L = (-1)^{N+1}, \quad a = 1, \dots, N. \quad (0.6)$$

The question of the completeness of the Bethe Ansatz, namely whether the set of solutions to (0.4), (0.5) or (0.6), is in one-to-one correspondence with the set of eigenvectors of $H_\Delta^{(N)}$ is tricky and remained open for quite a long time. A positive answer has been given, for the XXX chain, by Mukhin, Tarasov and Varchenko [29] provided that one agrees to slightly change the perspective and to characterise states in terms of polynomial solutions to an appropriate T – Q equations. Also, completeness was established for certain generic inhomogeneous variants of the XXZ chain [31].

Independently of completeness issues, there are numerous other questions related with the study of the equations (0.4)-(0.6). One relates to identifying the solution of (0.4), (0.5) or (0.6), depending on the value of Δ , giving rise to $H_\Delta^{(N)}$'s ground state, *viz.* the eigenvector associated with the lowest eigenvalue. The answer has been obtained by Yang-Yang in [36]. By applying a variant of the Perron-Frobenius theorem, Yang and Yang showed that $H_\Delta^{(N)}$ admits a unique ground state. In order to identify the roots describing the ground state, it is convenient to rewrite the Bethe equations in their logarithmic form

$$\frac{1}{2\pi} p(\lambda_a) - \frac{1}{2\pi L} \sum_{a=1}^N \theta(\lambda_a - \lambda_b) + \frac{N+1}{2L} = \frac{\ell_a}{L} \quad \text{with} \quad \ell_a \in \mathbb{Z}, \quad (0.7)$$

where for $-1 < \Delta = \cos(\zeta) < 1$, $\zeta \in]0; \pi[$

$$p(\lambda) = i \ln \left(\frac{\sinh(i\zeta/2 + \lambda)}{\sinh(i\zeta/2 - \lambda)} \right), \quad \theta(\lambda) = i \ln \left(\frac{\sinh(i\zeta + \lambda)}{\sinh(i\zeta - \lambda)} \right), \quad (0.8)$$

for $\Delta = 1$

$$p(\lambda) = i \ln \left(\frac{i/2 + \lambda}{i/2 - \lambda} \right) , \quad \theta(\lambda) = i \ln \left(\frac{i + \lambda}{i - \lambda} \right) , \quad (0.9)$$

and, for $\Delta = \cosh(\zeta) > 1$, $\zeta \in \mathbb{R}^+$,

$$p(\lambda) = \vartheta(\lambda, \zeta/2) , \quad \theta(\lambda) = \vartheta(\lambda, \zeta) \quad \text{with} \quad \vartheta(\lambda, \eta) = \int_0^\lambda \frac{\sinh(2\eta) d\mu}{\sin(\mu + i\eta) \sin(\mu - i\eta)} . \quad (0.10)$$

Yang and Yang proved that the ground state for $\Delta \geq -1$ is obtained from a real valued solution to (0.7) corresponding to the specific choice of integers $\ell_a = a$. More precisely, Yang and Yang were analysing a reparametrised version of the Bethe equations in terms of k_a variables, $\lambda_a = f_\Delta(k_a)$ for some explicit f_Δ . It was, in fact, the equation in terms of the variables k_a that has been initially obtained by Orbach [30]. Yang and Yang showed that the transformed counterpart of (0.7) admits a unique real valued solution when $-1 < \Delta \leq 0$. Furthermore, they showed that the transformed counterpart of (0.7) when $\ell_a = a$ admits, in fact, solutions for any $\Delta \geq -1$ and that, among these, there exists one such that $\Delta \mapsto k_a(\Delta)$ is continuous in $\Delta > -1$. Yang and Yang established that it is precisely this particular solution that gives rise to the Bethe roots parametrising the ground state of $H_\Delta^{(N)}$. I will refer, henceforth, to this solution as the ground state Bethe roots. Yang and Yang, however, did not prove the uniqueness of solutions to (0.7) for $\Delta > 0$ and $\ell_a = a$.

On top of the one parametrising the ground state of $H_\Delta^{(N)}$, the logarithmic Bethe Ansatz equations admit many other solutions upon varying the choices of integers ℓ_a . For instance, when the choice of the ground state integers is slightly perturbed, as

$$\ell_a = a \quad \text{for} \quad a \in \llbracket 1; N \rrbracket \setminus \{h_a\}_1^n \quad \ell_{h_a} = p_a \quad \text{with} \quad \begin{cases} h_a \in \llbracket 1; N \rrbracket \\ p_a \in \llbracket -M_\Delta; M_\Delta + N \rrbracket \setminus \llbracket 1; N \rrbracket \end{cases} \quad (0.11)$$

with M_Δ an integer depending on Δ , then the real-valued solution to the logarithmic Bethe Ansatz equations -if they exist- define so-called particle-hole excited states. One can also have complex valued solutions to the Bethe Ansatz equations, as already observed by Bethe [2]. See [4] for an extensive numerical analysis thereof in the case of small length XXX chains. In the present paper, I will *not* discuss the complex valued solutions.

From the point of view of practical applications one is mostly interested in the behaviour of observables attached to the model in the so-called thermodynamic limit $L \rightarrow +\infty$. Such observables can be the energy or momentum of an eigenstate or some correlation function. In practical situations this amounts to computing either the limit or the first few terms in the large- L asymptotic expansion of sums of the type

$$\frac{1}{L} \sum_{a=1}^N f(\lambda_j) \quad (0.12)$$

where f is some sufficiently regular function, $\{\lambda_a\}_1^N$ are the Bethe roots describing the ground state or some excited state "close" to it and the integer N labelling the spin sector to which the Bethe vector belongs is L dependent and grows with L in such a way that $N/L \rightarrow D \in [0; 1/2]$.

In [37], Yang and Yang affirmed that the limit exists in the case of the Bethe roots for the ground state and that, for any sufficiently regular function f , it holds

$$\lim_{L \rightarrow +\infty} \left\{ \frac{1}{L} \sum_{a=1}^N f(\lambda_j) \right\} = \int_{-q}^q f(\lambda) \rho(\lambda | q) \cdot d\lambda . \quad (0.13)$$

The pair $(q, \rho(\lambda \mid q))$ appearing above corresponds to the unique solution to the system of equations for the unknowns $(Q, \rho(\lambda \mid Q))$:

$$\rho(\lambda \mid Q) + \int_{-Q}^Q K(\lambda - \mu) \rho(\mu \mid Q) \cdot d\mu = \frac{1}{2\pi} p'(\lambda) \quad \text{and} \quad D = \int_{-Q}^Q \rho(\lambda \mid Q) \cdot d\lambda \quad (0.14)$$

where $D = \lim_{L \rightarrow +\infty} (N/L)$ and $K(\lambda) = \theta'(\lambda)/(2\pi)$, viz.

$$K(\lambda) = \begin{cases} \frac{\sinh(2\zeta)}{2\pi \sin(\mu + i\zeta) \sin(\mu - i\zeta)} & \text{for } \Delta > 1 \\ \frac{1}{\pi(1 + \lambda^2)} & \text{for } \Delta = 1 \\ \frac{\sin(2\zeta)}{2\pi \sinh(\mu + i\zeta) \sinh(\mu - i\zeta)} & \text{for } -1 < \Delta < 1 \end{cases} \quad (0.15)$$

Yang and Yang did prove that the system of equations on the pair of unknowns $(Q, \rho(\lambda \mid Q))$ does indeed admit a unique solution. They however did not prove the statement relative to the existence of the limit in the *lhs* of (0.13) nor its value given by the *rhs* of (0.13).

Numerous works, starting from the pioneering handlings of Hlten [13], did rely on the assumption that the limit of sums as in (0.12) exists and takes the form (0.13), be it when in the *lhs* there appear ground state Bethe roots or those describing a certain class of excited states above the ground state. In particular, such properties were used in the exact (but not rigorous in the sense introduced by Baxter [1]) calculations leading to characterising the ground state energy and spectrum of excitations of the infinite volume XXZ chain [5, 6, 13, 25, 26], testing the conformal structure of the spectrum of the XXZ chain [3, 16, 17, 24, 33, 34, 35], the algebraic Bethe Ansatz based calculations of the matrix elements of the reduced density matrix [21] and, more generally, ground state correlation functions in this model [15, 20] or of the large-volume behaviour of the matrix elements of local spin operators taken between two excited states close to the ground state [12, 14, 18, 19] so as to name a few. Despite its importance due to the mentioned multiple applications, the existence and form of the limit (0.13) was not proven in its full generality so far. The only work which did address this question was the one of Dorlas and Samsonov [10]. The two authors focused on the case of the ground state Bethe roots and proved that indeed (0.13) does hold for $-1 < \Delta \leq 0$ and for $\Delta > \Delta_0 > 1$ where Δ_0 was some explicit number. For $-1 < \Delta \leq 0$ they could build their proof on a generalisation of the convexity arguments that were invoked by Yang and Yang relatively to the existence and uniqueness of solutions to (0.7). Their argument was however limited to this regime since convexity does not hold anymore for $\Delta > 0$. The two authors also managed to prove the statement for $\Delta > \Delta_0 > 1$ by using the fixed point theorem for an auxiliary operator which was contracting for this range of the anisotropy.

One of the results of the present paper is the proof of (0.13) for all values of $\Delta > -1$ and for the class of real valued, particle-hole, solutions to the logarithmic Bethe equations (0.7) whose existence is established in Proposition 2.1. For simplicity, below, I only state the result in the case of the ground state Bethe roots. The general case can be found in Theorem 4.1.

Theorem. *Let $N, L \rightarrow +\infty$ in such a way that $N/L \rightarrow D$ with $D \in]0; 1/2]$. Let q be the unique solution to (0.14) subordinate to D and $\{\lambda_a\}_1^N$ correspond to the set of Bethe roots parametrising the ground state of $H_\Delta^{(N)}$. Then, given any bounded Lipschitz function f on \mathbb{R} , it holds*

$$\frac{1}{L} \sum_{a=1}^L f(\lambda_a) \rightarrow \int_{-q}^q f(\lambda) \rho(\lambda \mid q) \cdot d\lambda \quad (0.16)$$

In fact, this proposition is a corollary of a much stronger result established in the core of the paper : the existence of an all-order asymptotic expansion for the counting functions associated with such Bethe roots, see Theorems 3.5 and 3.4. The counting function contains all the fine details of the distribution of the λ_a 's, so that obtaining this asymptotic expansion goes much further than the simple limiting result (0.13). The idea of introducing the counting function as a way to study the large- L behaviour of a given solution to the Bethe equations goes back to the work of [9, 22]. The counting function formalism was further developed in the works [7, 8, 23, 24] what allowed to derive the first few terms of the large- L asymptotic expansion of the counting function associated with various configurations of Bethe roots associated with the XXZ chain. However, these derivations did build on various *ad hoc* hypothesis which, technically speaking, boil down to a densification property of the type (0.13) or a close variant thereof. In this paper I circumvent the use of such *ad hoc* hypothesis, hence bringing these formal asymptotic expansions to a rigorous level. It is important to stress that the method of proof introduced in the present paper neither relies on convexity arguments nor on fixed points theorem. The properties I use are rather general what makes the method applicable, *in principle*, to many other instances of quantum integrable models.

The paper is organised as follows. In Section 1, I review some properties of solutions to linear integral equations that are relevant to the problem and establish certain auxiliary results that are of interest to the analysis. In Section 2, I establish the solvability of the class of logarithmic Bethe equations of interest to the problem. Then, in Section 3, I establish the main result of the paper, namely the existence of the asymptotic expansion of the counting function. Section 4 is devoted to the applications of the results to various problems that arose earlier in the literature. In particular, the densification proposition is established there.

Appendix A contains an auxiliary result of interest to the analysis.

1 Some properties of solutions to linear integral equations

Let $J \subset \mathbb{R}$ be Lebesgue measurable. Let \mathcal{K}_J denote the integral operator on $L^2(J)$ acting as

$$\mathcal{K}_J[f](\lambda) = \int_J K(\lambda - \mu) f(\mu) d\mu . \quad (1.1)$$

Here $K(\lambda)$ corresponds to one of the three integral kernels given in (0.15), depending on the value of Δ . When $\Delta > 1$, I will always assume that $\text{diam}(J) \leq \pi$, where $\text{diam}(J)$ is the diameter of J . Throughout the paper, the dependence on Δ of the operators and integral kernels will be kept implicit since this would not bring more clarity to discussion while weighting down the intermediate formulae.

The purpose of this section is to recall some known facts about the operator $\text{id} + \mathcal{K}_J$ and the solutions to specific integral equation driven by it. In particular, I will discuss its invertibility for any J and review several properties of solutions of linear integral equation driven by this operator. Finally, I will prove an auxiliary result relative to the unique solvability of a non-linear problem driven by the operator \mathcal{K}_J . This unique solvability will appear crucial in a later stage of the analysis.

Throughout the paper, given $\alpha > 0$, I_α stands for the segment centred around 0

$$I_\alpha = [-\alpha ; \alpha] . \quad (1.2)$$

There are certain choices of the interval J which make the operator \mathcal{K}_J special, in that the linear integral equations driven by $\text{id} + \mathcal{K}_J$ can be solved in a closed form by using Fourier transforms or Fourier series. These choices correspond to $J = \mathbb{R}$ when $-1 < \Delta \leq 1$ and $J = [-\pi/2 ; \pi/2]$ when $\Delta > 1$. It is convenient to introduce a parameter ι labelling the range of integration I_ι corresponding to those special cases :

$$\iota = \begin{cases} \infty & \text{if } -1 < \Delta \leq 1 \\ \pi/2 & \text{if } \Delta > 1 \end{cases} \quad \text{so that} \quad I_\iota = \begin{cases} I_\infty = \mathbb{R} & \text{if } -1 < \Delta \leq 1 \\ I_{\pi/2} = [-\pi/2 ; \pi/2] & \text{if } \Delta > 1 \end{cases} . \quad (1.3)$$

It is a well known result that will be recalled in Lemma 1.3 that the operator $\text{id} + K_J$ is invertible for all J . The resolvent operator associated with K_J will be denoted by R_J . It is defined as the operator on $L^2(J)$ such that $(\text{id} - R_J)(\text{id} + K_J) = \text{id}$. The integral kernel of the resolvent will be denoted as $R_J(\lambda, \mu)$. It solves the linear integral equation

$$R_J(\lambda, \mu) + \int_J K(\lambda - \nu) R_J(\nu, \mu) d\nu = K(\lambda - \mu). \quad (1.4)$$

1.1 Some explicit solutions and their properties

1.1.1 The density of Bethe roots

The so-called density of Bethe roots is defined as the solution to the linear integral equations

$$(\text{id} + K_{I_q})[\rho(* | q)] = \frac{1}{2\pi} p'(\lambda). \quad (1.5)$$

The density can be computed in closed form when the support of integration is I_t .

Lemma 1.1. *The solution to the linear integral equation*

$$\begin{aligned} (\text{id} + K_{\mathbb{R}})[\rho_{\infty}](\lambda) &= \frac{1}{2\pi} p'(\lambda) \quad \text{for } -1 < \Delta \leq 1 \\ (\text{id} + K_{I_{\pi/2}})[\rho_{\pi/2}](\lambda) &= \frac{1}{2\pi} p'(\lambda) \quad \text{for } \Delta > 1 \end{aligned} \quad (1.6)$$

takes the form

$$\rho_{\infty}(\lambda) = \begin{cases} \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)} & \text{for } -1 < \Delta < 1 \\ \frac{1}{2 \cosh(\pi\lambda)} & \text{for } \Delta = 1 \end{cases} \quad \text{and} \quad \rho_{\pi/2}(\lambda) = \frac{1}{2\zeta} \sum_{n \in \mathbb{Z}} \frac{1}{\cosh\left[\frac{\pi}{\zeta}(n\pi - \lambda)\right]}. \quad (1.7)$$

The form of the solution, when $-1 < \Delta \leq 1$ is readily obtained by taking the Fourier transform of the linear integral equation. When $\Delta > 1$, one solves the linear integral equation by means of Fourier series expansions. This yields

$$\rho_{\pi/2}(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{2in\lambda}}{\cosh(n\zeta)}. \quad (1.8)$$

The expression (1.7) is obtained by applying the Poisson summation formula. These results appeared, for the first time, in [32, 37]. It is clear from (1.7) that both ρ_{∞} and $\rho_{\pi/2}$ are strictly positive functions.

1.1.2 The resolvent kernel

It is readily seen that $R_{I_t}(\lambda, \mu)$ only depends on the difference of its arguments. This integral kernel will be denoted below as $R(\lambda - \mu)$.

Lemma 1.2. *The solution $R(\lambda)$ to the equation $(\text{id} + K_{I_t})[R] = K$ is an even function for any $\Delta > -1$.*

- For $-1 < \Delta < 1$, R admits the Fourier transform representation

$$R(\lambda) = \int_{\mathbb{R}} \frac{\sinh[(\pi/2 - \zeta)k] e^{-ik\lambda}}{\cosh(\zeta k/2) \sinh[(\pi/2 - \zeta/2)k]} \frac{dk}{4\pi}, \quad \Delta = \cos(\zeta) \quad \text{with } \zeta \in]0; \pi[, \quad (1.9)$$

has the large λ estimates $R(\lambda) = O(e^{-\min(\frac{\pi}{\zeta}, \frac{2\pi}{\pi-\zeta})|\lambda|})$ and is a strictly positive functions when $0 < \Delta < 1$.

- For $\Delta = 1$, R admits the Fourier transform representation

$$R(\lambda) = \int_{\mathbb{R}} \frac{e^{-\frac{|k|}{2}} \cdot e^{-ik\lambda}}{\cosh(k/2)} \frac{dk}{4\pi}, \quad (1.10)$$

has the large- λ estimates $R(\lambda) = O(\lambda^{-2})$ and is strictly positive on \mathbb{R} .

- For $\Delta > 1$, R has the Fourier series expansion

$$R(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{2in\lambda} \cdot e^{-|n|\zeta}}{\cosh(n\zeta)}, \quad \Delta = \cosh(\zeta) \quad \text{with } \zeta > 0, \quad (1.11)$$

is π -periodic and strictly positive on \mathbb{R} .

The only non-trivial statement is the one relative to the signs of R . When $\Delta > 1$, using (1.8) and

$$\frac{p(\lambda)}{2\pi} = \sum_{n \in \mathbb{Z}} \frac{e^{-|n|\zeta} e^{2in\lambda}}{\pi} \quad \text{one gets} \quad R(\lambda) = \int_{-\pi/2}^{\pi/2} \rho_{\pi/2}(\lambda - \mu) p'(\mu) d\mu \quad (1.12)$$

from where strict positivity is manifest in virtue of (1.7). When $\Delta = 1$, one can recast the Fourier transform representation of R in the form $R(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \rho_{\infty}(\mu) p'(\lambda - \mu) d\mu$. The latter representation ensures strict positivity and readily yields the $O(\lambda^{-2})$ estimates for the decay of R at infinity. Finally, after some calculations, when $0 < \Delta < 1$ one recasts R as the convolution

$$R(\lambda) = \frac{\pi}{2\zeta(\pi - \zeta)} \int_{\mathbb{R}} \frac{\sin\left(\frac{\pi\zeta}{\pi - \zeta}\right)}{\cosh[\pi(\lambda - \mu)/\zeta] \sinh\left(\frac{\pi(\mu + i\zeta/2)}{\pi - \zeta}\right) \sinh\left(\frac{\pi(\mu - i\zeta/2)}{\pi - \zeta}\right)} \cdot d\mu \quad (1.13)$$

which produces a manifestly strictly positive function. ■

The representation (1.13) was first found in [37] while (1.12) is a straightforward application of the idea that leads to (1.13).

For the sake of further handlings, it is useful to introduce the integral operator L_J on $L^2(J)$ defined as

$$L_J[f](\lambda) = \int_J R(\lambda - \mu) f(\mu) d\mu. \quad (1.14)$$

It will be established in Lemma 1.3 that $\text{id} - L_J$ is invertible. The resolvent of this operator will be denoted \mathfrak{Q}_J and is defined by $(\text{id} - L_J)^{-1} = \text{id} + \mathfrak{Q}_J$.

1.2 General considerations

Lemma 1.3. *Let $J \subset \mathbb{R}$ be such that $0 < |J| < +\infty$ if $-1 < \Delta \leq 1$ and $\text{diam}(J) < \pi$ if $\Delta > 1$. Then, the operators $\text{id} + K_J$ and $\text{id} - L_J$ are both invertible. The integral kernel $R_J(\lambda, \mu)$ of the resolvent operator R_J satisfies to the bounds*

$$R(\lambda - \mu) < R_J(\lambda, \mu) < K(\lambda - \mu) < 0 \quad \text{for } -1 < \Delta < 0 \quad (1.15)$$

and

$$K(\lambda - \mu) > R_J(\lambda, \mu) > R(\lambda - \mu) > 0 \quad \text{for } \Delta > 0. \quad (1.16)$$

Furthermore, one has $\mathfrak{L}_{J^c}(\lambda, \mu) = R_J(\lambda, \mu)$, where $J^c = I_t \setminus J$.

Proof —

For any $g \in L^\infty(J)$, one has the bounds

$$\|K_J[g]\|_{L^\infty(\mathbb{R})} \leq \|g\|_{L^\infty(J)} \cdot \sup_{\lambda \in \mathbb{R}} \int_{J+\lambda} |K(\mu)| d\mu \leq \|g\|_{L^\infty(J)} \cdot \sup_{I \in \mathcal{I}_\Delta} \|K\|_{L^1(I)} \leq \|g\|_{L^\infty(J)} \cdot \|K\|_{L^1(I_{|J|/2})} \quad (1.17)$$

where $I_{|J|/2} = [-|J|/2; |J|/2]$ and

$$\mathcal{I}_\Delta = \{I \subset \mathbb{R} : |I| = |J|\} \quad \text{if } -1 < \Delta \leq 1 \quad \text{and} \quad \mathcal{I}_\Delta = \{I \subset [-\pi/2; \pi/2] : |I| = |J|\} \quad \text{when } \Delta > 1. \quad (1.18)$$

The second bound is trivial for $-1 < \Delta \leq 1$ and it holds for $\Delta > 1$ since by π -periodicity of K one can reduce the integration along an interval of diameter at most π into one over an appropriate subinterval of $[-\pi/2; \pi/2]$. The bound holds since $|K|$ is even and decreasing on \mathbb{R}^+ , resp. $[0; \pi/2]$, when $-1 < \Delta \leq 1$, resp. $\Delta > 1$.

Now due to

$$\int_{\mathbb{R}} K(\mu) d\mu = \frac{\pi - 2\zeta}{\pi} \quad \text{if } -1 < \Delta \leq 1 \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} K(\mu) d\mu = 1 \quad \text{if } \Delta > 1 \quad (1.19)$$

the hypothesis on J ensure that $\|K\|_{L^1(I_{|J|/2})} < 1$ and thus that the Neumann series

$$R_J(\lambda, \mu) = K(\lambda - \mu) + \sum_{n \geq 1} (-1)^{n-1} \int_{J^n} K(\lambda - \nu_1) \prod_{a=1}^{n-1} K(\nu_a - \nu_{a+1}) \cdot K(\nu_n - \mu) \cdot d^n \nu \quad (1.20)$$

for the resolvent kernel converges uniformly on \mathbb{R}^2 . This establishes the invertibility of $(\text{id} + K_J)$. The one of $(\text{id} - L_J)$ is proven analogously using the properties of the resolvent $R(\lambda)$ established in Lemma 1.2.

When $-1 < \Delta < 0$ the Neumann series for $R_J(\lambda, \mu)$ is a sum of strictly negative terms. Hence the upper bounds given in (1.15). For $\Delta > 0$, following Yang and Yang [37] it is enough to observe that the integral equation for $R_J(\lambda, \mu)$ can be recast as

$$(\text{id} + K_{I_t})[R_J(*, \mu)](\lambda) - K_{J^c}[R_J(*, \mu)](\lambda) = K(\lambda - \mu) \quad \text{i.e.} \quad (\text{id} - L_{J^c})[R_J(*, \mu)](\lambda) = R(\lambda - \mu) \quad (1.21)$$

where I remind that $J^c = I_t \setminus J$. Since $R(\lambda) > 0$ for $\Delta > 0$, the Neumann series[‡] for the resolvent kernel $\mathfrak{L}_{J^c}(\lambda, \mu)$ consists of strictly positive terms what entails $R_J(\lambda, \mu) > R(\lambda - \mu)$. The upper bound follows from $K(\lambda - \mu) \geq 0$, $R_J(\lambda, \mu) \geq 0$ and $R_J(\lambda, \mu) = K(\lambda - \mu) - K_J[R_J(*, \mu)](\lambda)$. Finally, the equality $\mathfrak{L}_{J^c}(\lambda, \mu) = R_J(\lambda, \mu)$ between the integral kernels follows from the fact that $\lambda \mapsto \mathfrak{L}_{J^c}(\lambda, \mu)$ is the unique solution to the linear integral equation appearing to the right of (1.21). ■

‡. Its convergence follows from similar bounds and properties as those used for the one associated with $K(\lambda)$

Lemma 1.4. Let $\Delta > 0$ and $J^{(\pm)}$ be two bounded and disjoint subsets of I_L . Let $[R_{J^{(+)}}]_{J^{(-)}}$ be the integral operator on $L^2(J^{(-)})$ acting as

$$[R_{J^{(+)}}]_{J^{(-)}}[f](\lambda) = \int_{J^{(-)}} R_{J^{(+)}}(\lambda, \mu) f(\mu) \cdot d\mu. \quad (1.22)$$

Then, the operator $(\text{id} - [R_{J^{(+)}}]_{J^{(-)}})$ is invertible. Its resolvent operator $[\mathcal{R}_{J^{(+)}}]_{J^{(-)}}$ defined by

$$(\text{id} - [R_{J^{(+)}}]_{J^{(-)}})(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}}) = \text{id} \quad (1.23)$$

has a strictly positive resolvent kernel : $[\mathcal{R}_{J^{(+)}}]_{J^{(-)}}(\lambda, \mu) > 0$ for any $\lambda, \mu \in \mathbb{R}$.

Proof —

For any $g \in L^\infty(J^{(-)})$ it holds

$$\begin{aligned} \|[R_{J^{(+)}}]_{J^{(-)}}[g]\|_{L^\infty(\mathbb{R})} &\leq \|g\|_{L^\infty(J^{(-)})} \cdot \sup_{\lambda \in \mathbb{R}} \int_{J^{(-)}} |R_{J^{(+)}}(\lambda, \mu)| d\mu \\ &\leq \|g\|_{L^\infty(J^{(-)})} \cdot \sup_{\lambda \in \mathbb{R}} \int_{J^{(+)+\lambda}} |K(\mu)| d\mu \leq \|g\|_{L^\infty(J^{(-)})} \cdot \|K\|_{L^1(I_{|J^{(+)|/2})} \end{aligned} \quad (1.24)$$

where the second bound follows from the inequality (1.16). The rest of the proof is carried out analogously to the one of Lemma 1.3. ■

1.3 The magnetic Fermi boundaries

When studying the observables of the XXZ chain in the thermodynamic limit $N/L \rightarrow D$, one naturally ends up with operators $\text{id} + \mathbb{K}_{I_Q}$ where the endpoint Q is one of the unknowns of the problem. The endpoint $q(D)$ which will be pertinent for describing the thermodynamics at given D and the so-called dressed momentum $p(\lambda | q(D))$ describing the momentum carried by an individual excitation over the model's ground state arise as the solution to the below problem for two unknowns $(Q, p(\lambda | Q))$:

$$(\text{id} + \mathbb{K}_{I_Q})[p(\cdot | Q)](\lambda) = \frac{p(\lambda)}{2\pi} - \frac{D}{4\pi} [\theta(\lambda - Q) + \theta(\lambda + Q)] \quad \text{and} \quad p(Q | Q) = \frac{D}{2}. \quad (1.25)$$

It was shown by Yang and Yang [37] that for any $D \in [0; 1/2]$ there exists a unique $q(D) \in \mathbb{R}^+$ such that $(q(D), p(\lambda | q(D)))$ solves the problem given above. Furthermore, for $\Delta > 1$ one even has $q(D) \in [0; \pi/2]$. The proof is relatively straightforward and builds on the fact that $Q \mapsto \int_{-Q}^Q p'(\lambda | Q) d\lambda$ is an increasing function of Q , c.f. [11] for some details. The map $D \mapsto q(D)$ is smooth and strictly increasing diffeomorphism from $[0; 1/2]$ onto $[0; +\infty]$.

In the following, I will simply denote by q the endpoint of integration solving (1.25). This endpoint will be referred to as the magnetic Fermi boundary.

It turns out that in the intermediate steps of the analysis, it will become necessary to consider a slightly more general problem to (1.25), namely one for three unknowns $(Q_L, Q_R, f(\lambda | Q_L, Q_R))$

$$(\text{id} + \mathbb{K}_{[Q_L; Q_R]})[f(\cdot | Q_L, Q_R)](\lambda) = \frac{p(\lambda)}{2\pi} - \frac{D}{4\pi} \cdot [\theta(\lambda - Q_R) + \theta(\lambda - Q_L)] \quad (1.26)$$

and

$$f(Q_R | Q_L, Q_R) = \frac{D}{2} \quad f(Q_L | Q_L, Q_R) = -\frac{D}{2} \quad (1.27)$$

with $Q_L < Q_R$, and the additional constraint $|Q_R - Q_L| < \pi$ if $\Delta > 1$.

Proposition 1.1. *Given any $D \in [0; 1/2]$, the problem (1.26)-(1.27) admits the unique solution $(q, -q, p(\lambda | q))$, where $(q, p(\lambda | q))$ corresponds to the unique solution to the problem (1.25).*

Proof —

It is evident that $(q, -q, p(\lambda | q))$ provides one with a solution to the given problem. Hence, it remains to prove uniqueness. Let $(Q_L, Q_R, f(\lambda | Q_L, Q_R))$ be a solution to (1.26)-(1.27). It is convenient to distinguish between the case where both Q_L and Q_R are infinite, one of them is or both are bounded. Note that the first two situations can only arise when $-1 < \Delta \leq 1$.

Both endpoints are infinite

First assume that $Q_L = -\infty$ and $Q_R = +\infty$. Then,

$$f(\lambda | -\infty, +\infty) = p(\lambda | +\infty) \equiv \int_0^\lambda \rho_\infty(\lambda) d\lambda \quad \Rightarrow \quad p(\pm\infty | +\infty) = \pm \frac{1}{4} \quad (1.28)$$

so that $D = 1/2$ and either this cannot be or one simply recovers the solution to the problem (1.25).

One of the endpoints is infinite

By symmetry, it is enough to deal with the case where Q_L is bounded while $Q_R = +\infty$. For simplicity, denote simply the solution by $f(\lambda)$. An integration by parts ensures that $(\text{id} + \mathbb{K}_{[Q_L; +\infty)}][f'] = p'/2\pi$. Then, it follows from the bounds (1.15) and (1.16) that

$$\frac{p'(\lambda)}{2\pi} < f'(\lambda) < \rho_\infty(\lambda) \quad \text{for} \quad -1 < \Delta < 0 \quad \text{and} \quad \frac{p'(\lambda)}{2\pi} > f'(\lambda) > \rho_\infty(\lambda) \quad \text{for} \quad 0 < \Delta \leq 1. \quad (1.29)$$

These bounds ensure that f is bounded on \mathbb{R} and strictly increasing. As such it admits a limit at $\pm\infty$. Hence

$$|K(\lambda - \mu)f(\mu)| \leq Ce^{\lambda - \mu} \quad \text{for} \quad \lambda \leq Q_L \quad \text{so that} \quad \lim_{\lambda \rightarrow -\infty} \left\{ \mathbb{K}_{[Q_L; +\infty)}[f](\lambda) \right\} = 0 \quad (1.30)$$

by dominated convergence. Thence

$$\lim_{\lambda \rightarrow -\infty} f(\lambda) = -\frac{D}{2} - \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - D \right). \quad (1.31)$$

Furthermore, due to the upper bound in (1.29) and $\lim_{\lambda \rightarrow +\infty} f(\lambda) = D/2$ it holds, for $\lambda \geq Q_L$

$$|f(\lambda) - D/2| \leq Ce^{-\min(2, \frac{\pi}{\zeta})\lambda} \leq Ce^{-\lambda} \quad (1.32)$$

so that

$$\left| \mathbb{K}_{[Q_L; +\infty)}[f - D/2](\lambda) \right| \leq C \int_{\mathbb{R}} |K(\mu)| e^{-(\lambda + \mu)} \cdot d\mu \xrightarrow{\lambda \rightarrow +\infty} 0. \quad (1.33)$$

Thus recasting the integral equation for f in the form

$$f(\lambda) + \int_{Q_L}^{+\infty} K(\lambda - \mu) \left[f(\mu) - \frac{D}{2} \right] \cdot d\mu = \frac{p(\lambda)}{2\pi} - \frac{D}{2\pi} \theta(\lambda - Q_L). \quad (1.34)$$

Taking the $\lambda \rightarrow +\infty$ limit of the above equation yields an equation for D which implies that $D = 1/2$. In its turn, due to (1.31), this implies that $\lim_{\lambda \rightarrow -\infty} [f(\lambda)] = -D/2$ hence contradicting $f(Q_L) = -D/2$ since f is strictly increasing on \mathbb{R} by (1.29).

Both endpoints are bounded

In this case, it is convenient to introduce

$$Q = \frac{Q_R - Q_L}{2}, \quad a = \frac{Q_R + Q_L}{2} \quad \text{and} \quad \widetilde{f}(\lambda) = f(\lambda + a | Q_L, Q_R). \quad (1.35)$$

One gets that $\widetilde{f}(\lambda)$ satisfies

$$(\text{id} + \mathbb{K}_{I_Q})[\widetilde{f}] = \frac{p(\lambda + a)}{2\pi} - \frac{D}{4\pi} \cdot [\theta(\lambda - Q) + \theta(\lambda + Q)] \quad \text{and} \quad \widetilde{f}(Q) = -\widetilde{f}(-Q) = \frac{D}{2}. \quad (1.36)$$

The function \widetilde{f} can be uniquely decomposed as $\widetilde{f}(\lambda) = \widetilde{f}_i(\lambda) + \widetilde{f}_p(\lambda)$ where \widetilde{f}_p is even and \widetilde{f}_i is odd. It is readily seen that the functions \widetilde{f}_p and \widetilde{f}_i satisfy the linear integral equations

$$(\text{id} + \mathbb{K}_{I_Q}) \left[\begin{pmatrix} \widetilde{f}_i \\ \widetilde{f}_p \end{pmatrix} \right](\lambda) = \frac{1}{4\pi} \begin{pmatrix} p(\lambda + a) + p(\lambda - a) - D \cdot [\theta(\lambda - Q) + \theta(\lambda + Q)] \\ p(\lambda + a) - p(\lambda - a) \end{pmatrix} \quad (1.37)$$

and are subject to the constraints

$$\widetilde{f}_p(Q) = 0 \quad \text{and} \quad \widetilde{f}_i(Q) = \frac{D}{2}. \quad (1.38)$$

The integral operator appearing above acts component-wise on the entries of the vector. Consider the equation satisfied by the even part \widetilde{f}_p . It will be shown that the constraint $\widetilde{f}_p(Q) = 0$ can only be satisfied if $a = 0$. Once this is established, then (1.37) reduces to (1.25), what ensures uniqueness. In the course of doing so, one should treat the two regimes $-1 < \Delta \leq 0$ and $\Delta > 0$ separately due to the change in the sign of the integral kernel $K(\lambda - \mu)$.

• $-1 < \Delta \leq 0$

By (1.15), the integral kernel of the resolvent operator R_{I_Q} to $\text{id} + \mathbb{K}_{I_Q}$ is such that $R_{I_Q}(\lambda, \mu) \leq 0$. Thus, due to (1.37), \widetilde{f}_p can be represented as

$$\widetilde{f}_p(\lambda) = \frac{p(\lambda + a) - p(\lambda - a)}{4\pi} - \int_{-Q}^Q R_{I_Q}(\lambda, \mu) [p(\mu + a) - p(\mu - a)] \cdot \frac{d\mu}{4\pi}. \quad (1.39)$$

Since

$$p(\lambda + a) - p(\lambda - a) = \int_{\lambda-a}^{\lambda+a} p'(\mu) \cdot d\mu \quad (1.40)$$

has the same sign as a , it follows that, for any $\lambda \in \mathbb{R}^+$,

$$\begin{aligned} \text{if } a > 0 \quad \text{then} \quad \widetilde{f}_p(\lambda) &\geq \frac{1}{4\pi}[\mathfrak{p}(\lambda + a) - \mathfrak{p}(\lambda - a)] > 0 \\ \text{if } a < 0 \quad \text{then} \quad \widetilde{f}_p(\lambda) &\leq \frac{1}{4\pi}[\mathfrak{p}(\lambda + a) - \mathfrak{p}(\lambda - a)] < 0 \end{aligned} \quad (1.41)$$

Either of the two are inconsistent with the constraint $\widetilde{f}_p(Q) = 0$. Therefore, necessarily, $a = 0$ so that the problem reduces to (1.25) and on that account is uniquely solvable.

• $\Delta > 0$

We start by observing that the solution $\phi_{\iota,a}$, c.f. (1.3) for the definition of ι , to the linear integral equation

$$(\text{id} + K_{I_\iota})[\phi_{\iota,a}](\lambda) = \frac{\mathfrak{p}(\lambda + a) - \mathfrak{p}(\lambda - a)}{2\pi} \quad \text{takes the form} \quad \phi_{\iota,a}(\lambda) = \int_{\lambda-a}^{\lambda+a} \rho_\iota(\mu) \cdot d\mu. \quad (1.42)$$

This can be seen as follows. When $\iota = \infty$, the decay at infinity of the resolvent kernel $R(\lambda - \mu)$ and the boundedness of the driving term ensure that $\phi_{\infty,a}$ is bounded on \mathbb{R} . When $\iota = \pi/2$, it is readily inspected that $\phi_{\pi/2,a}$ is π -periodic. Thus, for any ι either because the boundary terms vanish ($\iota = \infty$) or cancel out ($\iota = \pi/2$), taking the derivative of the linear integral equation in (1.42) and then integrating by parts one gets a linear integral equation satisfied by $\phi'_{\iota,a}$ whose solution reads $\phi'_{\iota,a}(\lambda) = \rho_\iota(\lambda + a) - \rho_\iota(\lambda - a)$. The integral representation for $\phi_{\iota,a}$ then follows from the fact that it is an even function.

\widetilde{f}_p can be readily continued by means of the linear integral equation to the real axis. Then, building on the identity

$$(\text{id} + K_{I_Q})[\widetilde{f}_p](\lambda) = (\text{id} + K_{I_\iota})[\widetilde{f}_p](\lambda) - K_{I_Q^c}[\widetilde{f}_p](\lambda) \quad \text{with} \quad I_Q^c = I_\iota \setminus I_Q \quad (1.43)$$

it is easily seen, in virtue of (1.42), that \widetilde{f}_p satisfies the linear integral equation

$$(\text{id} - L_{I_Q^c})[\widetilde{f}_p](\lambda) = \frac{1}{2}\phi_{\iota,a}(\lambda) \quad (1.44)$$

where $L_{I_Q^c}$ is as defined in (1.14). As a consequence, it holds,

$$\widetilde{f}_p(\lambda) = \frac{1}{2}\phi_{\iota,a}(\lambda) + \int_{I_Q^c} \mathfrak{L}_{I_Q^c}(\lambda, \mu)\phi_{\iota,a}(\mu) \cdot \frac{d\mu}{2}. \quad (1.45)$$

As a consequence, for any $\lambda \in \mathbb{R}^+$,

$$\begin{aligned} \text{if } a > 0 \quad \text{then} \quad \widetilde{f}_p(\lambda) &\geq \frac{1}{2}\phi_{\iota,a}(\lambda) > 0 \quad \text{for } \lambda > 0 \\ \text{if } a < 0 \quad \text{then} \quad \widetilde{f}_p(\lambda) &\leq \frac{1}{2}\phi_{\iota,a}(\lambda) < 0 \quad \text{for } \lambda > 0 \end{aligned} \quad (1.46)$$

Again, either of the two are inconsistent with the constraint $\widetilde{f}_p(Q) = 0$ so that, necessarily, $a = 0$. ■

1.4 Auxiliary functions

In this last subsection I introduce a few auxiliary special functions that will arise in a later stage of the analysis. These functions are defined as solutions to linear integral equations driven by $\text{id} + \mathbb{K}_{I_Q}$ and thus depend on a free parameter Q .

The dressed phase $\varphi(\lambda, \mu | Q)$ is defined as the solution to the linear integral equation

$$(\text{id} + \mathbb{K}_{I_Q})[\varphi(*, \mu | Q)](\lambda) = \frac{\theta(\lambda - \mu)}{2\pi}. \quad (1.47)$$

Above, $*$ denotes the running argument of the function on which the integral operator acts. The dressed charge $Z(\lambda | Q)$ is defined as the solution to the linear integral equation

$$(\text{id} + \mathbb{K}_I)[Z(* | Q)](\lambda) = 1. \quad (1.48)$$

The dressed energy $\varepsilon(\lambda | Q)$ is defined as the solution to the linear integral equation

$$(\text{id} + \mathbb{K}_I)[\varepsilon(* | Q)](\lambda) = \varepsilon(\lambda) \quad \text{with} \quad \varepsilon(\lambda) = h - 2J\chi_\Delta \cdot \mathfrak{p}'(\lambda) \quad (1.49)$$

where the constant χ_Δ depends on the anisotropy as

$$\begin{aligned} \chi_\Delta &= \sin \zeta & \text{for } \Delta &= \cos(\zeta) & 0 < \zeta < \pi \\ \chi_\Delta &= 1 & \text{for } \Delta &= 1 \\ \chi_\Delta &= \sinh(\zeta) & \text{for } \Delta &= \cosh(\zeta) & \zeta > 0 \end{aligned}. \quad (1.50)$$

Finally, the thermodynamic counting function is defined in terms of the dressed momentum $p(\lambda | q)$ as

$$\xi_0(\lambda | q) = p(\lambda | q) + \frac{D}{2}. \quad (1.51)$$

Lemma 1.5. *The dressed phase is related to the dressed charge as*

$$\varphi(\lambda, Q | Q) - \varphi(\lambda, -Q | Q) + 1 = Z(\lambda | q) \quad \text{and} \quad 1 + \varphi(Q | Q) - \varphi(-Q | Q) = \frac{1}{Z(Q | Q)} \quad (1.52)$$

and also satisfies

$$\partial_\lambda \varphi(\lambda, \mu | Q) = R_{I_Q}(\lambda, \mu) + R_{I_Q}(\lambda, Q)\varphi(Q, \mu | Q) - R_{I_Q}(\lambda, -Q)\varphi(-Q, \mu | Q). \quad (1.53)$$

Also, the thermodynamic counting function satisfies $\xi'_0(\lambda | q) = \rho(\lambda | q) > 0$ and thus is a strictly increasing diffeomorphism from

$$\mathbb{R} \quad \text{onto} \quad \xi_0(\mathbb{R} | q) = \left[-\frac{\pi - \zeta}{\pi}(\frac{1}{2} - D); \frac{\pi - \zeta}{\pi}(\frac{1}{2} - D) + D \right] \quad (1.54)$$

for $-1 < \Delta \leq 1$ and from \mathbb{R} onto \mathbb{R} when $\Delta > 1$.

The proof of most statements is rather straightforward. In fact, the only non-trivial identity corresponds to the relationship with $Z^{-1}(Q | Q)$. The latter was established by Korepin and Slavnov [27] and I refer to their paper for more details.

For magnetic fields below the critical field

$$h_c = \begin{cases} 8J \cos^2(\zeta/2) & -1 < \Delta = \cos(\zeta) \leq 1 \quad \zeta \in [0; \pi[\\ 8J \cosh^2(\zeta/2) & \Delta = \cosh(\zeta) > 1 \quad \zeta > 0 \end{cases} \quad (1.55)$$

the XXZ chain is an antiferromagnet. Furthermore, when $\Delta > 1$, $0 < h < h_c^{(L)}$ the model has a mass gap with $h_c^{(L)} = 8J \sinh^2(\zeta/2)$ whereas it is massless for $h_c^{(L)} \leq h < h_c$.

Proposition 1.2. Assume that the magnetic field h satisfies to the bounds

$$0 < h < h_c \quad \text{for} \quad -1 < \Delta < 1 \quad \text{and} \quad h_c^{(L)} \leq h < h_c \quad \text{for} \quad \Delta > 1. \quad (1.56)$$

Then, there exists a unique solution Q_F , called the Fermi boundary, such that $\varepsilon(Q_F | Q_F) = 0$. Furthermore, it holds

$$\varepsilon(Q | Q) < 0 \quad \text{for} \quad Q \in]-Q_F; Q_F[\quad \text{and} \quad \varepsilon(Q | Q) > 0 \quad \text{for} \quad Q \in I_{Q_F}^c. \quad (1.57)$$

This result has been established in [11] when $-1 < \Delta < 1$ and the technique of proof can be readily extended to the regime $\Delta \geq 1$. I refer the interested reader to that paper for more details.

2 Solvability and boundedness

2.1 Existence of solutions to the logarithmic Bethe equations

Proposition 2.1.

- $-1 < \Delta \leq 1$

Let $\zeta \in [0; \pi[$ parametrise $\Delta = \cos(\zeta)$ and

$$h_1 < \dots < h_n, \quad h_a \in \llbracket 1; N \rrbracket \quad , \quad p_1 < \dots < p_n, \quad p_a \in \mathbb{Z} \setminus \llbracket 1; N \rrbracket, \quad (2.1)$$

be ordered integers such that

$$\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N-1}{L} \right) > \frac{p_n - N}{L} \quad , \quad \frac{p_1 - 1}{L} > -\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N-1}{L} \right) \quad \text{and} \quad \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} \right) \geq \frac{n}{L}. \quad (2.2)$$

Define the set of integers $\{\ell_a\}_1^N$ by

$$\ell_a = a \quad \text{for} \quad a \in \llbracket 1; N \rrbracket \setminus \{h_1, \dots, h_n\} \quad \text{and} \quad \ell_{h_a} = p_a \quad \text{for} \quad a = 1, \dots, n. \quad (2.3)$$

Then, the system of logarithmic Bethe equations (0.7) with \mathfrak{p} and θ as defined in (0.8) or (0.9) admits a solution such that all Bethe roots $\{\lambda_a\}_1^N$ are real. Furthermore, for $\zeta \in [\pi/2; \pi[$, this solution is unique.

- $\Delta > 1$

Let $\zeta \in]0; +\infty[$ parametrise $\Delta = \cosh(\zeta)$ and $h_a \in \llbracket 1; N \rrbracket$ and $p_a \in \mathbb{Z} \setminus \llbracket 1; N \rrbracket$ be any ordered integers, viz. $h_1 < \dots < h_n$ and $p_1 < \dots < p_n$. Then, the system of logarithmic Bethe equations (0.7) with \mathfrak{p} and θ as defined in (0.10) admits a solution such that all Bethe roots $\{\lambda_a\}_1^N$ are real.

Note that the proposition only stipulates the existence of solutions in the case of general $\Delta > -1$. Uniqueness only holds, *a priori*, for $0 \geq \Delta > -1$. Furthermore, when $\Delta > 1$, the proposition only states that the λ_a 's are real and says nothing about the domain to which they belong. The main peculiarity of the $\Delta > 1$ regime is that two solutions of the logarithmic Bethe equations which differ by translations of π , namely $\lambda_a = \lambda'_a + m_a \pi$ for $a = 1, \dots, N$ and some $m_a \in \mathbb{Z}$ define *equivalent* solution to the Bethe equations. Hence, distinct sets of integers ℓ_a and ℓ'_a do not necessarily allow one, when $\Delta > 1$, to distinguish between inequivalent solutions. So as to deal with only one representative one should, *in fine* only focus on the solutions belonging to some fixed interval of length π , say $] -\pi/2; \pi/2]$. This is, however, something that should be done *a posteriori*, after having built the solutions.

The proof of this statement basically follows Yang-Yang's argument for the ground state Bethe roots when $-1 < \Delta < 0$. The idea consist in introducing a function $\mu \mapsto S_\ell(\mu)$ on \mathbb{R}^N such that the logarithmic Bethe Ansatz

equations correspond to the conditions for a local minimum of S_ℓ . The main new observation introduced here is that even though the function S_ℓ is *not* convex for $\Delta > 0$, it still blows up at ∞ provided that (2.2) holds, and hence admits a minimum.

Proof —

- $-1 < \Delta \leq 1$

Following the reasoning of Yang and Yang, one introduces the function

$$S_\ell(\mu) = \sum_{a=1}^N \frac{P(\mu_a)}{2\pi} - \frac{1}{4\pi L} \sum_{a,b=1}^N \Theta(\mu_a - \mu_b) - \sum_{a=1}^N \mu_a \frac{n_a}{L} \quad \text{with} \quad n_a = \ell_a - \frac{N+1}{2} \quad (2.4)$$

defined in terms of

$$P(\lambda) = \int_0^\lambda p(\mu) d\mu = (\pi - \zeta)|\lambda| + \begin{cases} O(1) \\ O(\ln |\lambda|) \end{cases} \quad \text{and} \quad \Theta(\lambda) = \int_0^\lambda \theta(\mu) d\mu = (\pi - 2\zeta)|\lambda| + \begin{cases} O(1) \\ O(\ln |\lambda|) \end{cases} . \quad (2.5)$$

The remainders appearing above are such that the first line holds pointwise in $\zeta \in]0; \pi[$ while the second line holds for $\zeta = 0$. This convention will be carried on until the end of this proof.

It is easy to see that the logarithmic Bethe equations (0.7) appear as conditions for the existence of a critical point λ of $S_\ell(\mu)$:

$$\frac{\partial}{\partial \mu_a} \cdot S_\ell(\mu) \Big|_{\lambda=\mu} = \frac{p(\lambda_a)}{2\pi} - \frac{1}{2\pi L} \sum_{b=1}^N \theta(\lambda_a - \lambda_b) - \frac{n_a}{L} . \quad (2.6)$$

Thus, it is enough to show that $S_\ell(\mu) \rightarrow +\infty$ as $\mu \rightarrow +\infty$ so as to ensure the existence of a minimum of $S_\ell(\mu)$ and hence the existence of a solution to the logarithmic Bethe equations.

It appears convenient to study the behaviour of $S_\ell(\mu)$ when μ goes to infinity along a ray. In such a situation, there exists $\sigma \in \mathfrak{S}_N$ such that $\mu_\sigma = (\mu_{\sigma(1)}, \dots, \mu_{\sigma(N)})$ goes to infinity as

$$\mu_\sigma = t\mathbf{v} \quad \text{with} \quad \mathbf{v} = (\underbrace{v_1, \dots, v_1}_{\alpha_1 \text{ terms}}, \dots, \underbrace{v_s, \dots, v_s}_{\alpha_s \text{ terms}}) \quad v_1 < \dots < v_r \leq 0 < v_{r+1} < \dots < v_s . \quad (2.7)$$

There $t \rightarrow +\infty$ and $\mathbf{v} \in \mathbb{S}^N$, the N -dimensional sphere. Given μ as above, when $t \rightarrow +\infty$, it holds

$$S_\ell(\mu) = -t \frac{\pi - \zeta}{2\pi} \sum_{\ell=1}^r \alpha_\ell v_\ell + t \frac{\pi - \zeta}{2\pi} \sum_{\ell=r+1}^s \alpha_\ell v_\ell - t \frac{\pi - 2\zeta}{2\pi L} \sum_{j < \ell} \alpha_j \alpha_\ell (v_\ell - v_j) - \frac{t}{L} \sum_{\ell=1}^s v_\ell \bar{n}_\ell + \begin{cases} O(1) \\ O(\ln t) \end{cases} . \quad (2.8)$$

Here, we have set

$$\bar{n}_\ell = \sum_{j=a_{\ell-1}+1}^{a_\ell} n_{\sigma(j)} \quad \text{and} \quad a_\ell = \sum_{p=1}^{\ell} \alpha_p , \quad a_0 = 0 . \quad (2.9)$$

Therefore, using that

$$\sum_{j < \ell}^s \alpha_j \alpha_\ell (v_\ell - v_j) = \sum_{\ell=2}^s \alpha_\ell v_\ell a_{\ell-1} - \sum_{j=1}^{s-1} \alpha_j v_j (N - a_j) , \quad (2.10)$$

one can recast the large t asymptotics of $S_\ell(\boldsymbol{\mu})$ as

$$S_\ell(\boldsymbol{\mu}) = t \left\{ \sum_{\ell=1}^{r-1} (v_\ell - v_{\ell+1}) \cdot \left(\sum_{j=1}^{\ell} \tau_j^{(-)} \right) + v_r \cdot \left(\sum_{j=1}^r \tau_j^{(-)} \right) \right. \\ \left. + \sum_{\ell=2+r}^s (v_\ell - v_{\ell-1}) \cdot \left(\sum_{j=\ell}^s \tau_j^{(+)} \right) + v_{r+1} \cdot \left(\sum_{j=1+r}^s \tau_j^{(+)} \right) \right\} + \begin{cases} O(1) \\ O(\ln t) \end{cases} , \quad (2.11)$$

where

$$\tau_\ell^{(\pm)} = \alpha_\ell \left\{ \pm \frac{\pi - \zeta}{2\pi} + \frac{N(\pi - 2\zeta)}{2\pi L} - \frac{a_{\ell-1} + a_\ell}{2L} \frac{\pi - 2\zeta}{\pi} \right\} - \frac{\bar{n}_\ell}{L} . \quad (2.12)$$

One will have $S_\ell(\boldsymbol{\mu}) \rightarrow +\infty$ for any $\boldsymbol{\mu} \rightarrow \infty$ provided that

$$\sum_{j=1}^{\ell} \tau_j^{(-)} < 0 \text{ for all } \ell \in \llbracket 1 ; r \rrbracket \quad \text{and} \quad \sum_{j=\ell}^s \tau_j^{(+)} > 0 \text{ for all } \ell \in \llbracket r+1 ; s \rrbracket \quad (2.13)$$

this for all choices of point $\mathbf{v} \in \mathbb{S}^N$, viz. for all admissible values of r, s and of the α_a 's. These equation impose constraints on the integers n_a . Indeed, using that

$$\sum_{p=1}^{\ell} \alpha_p (a_{p-1} + a_p) = a_\ell^2 , \quad \sum_{p=\ell}^s \alpha_p (a_{p-1} + a_p) = 2N\tilde{a}_\ell - \tilde{a}_\ell^2 \quad \text{with} \quad \tilde{a}_\ell = \sum_{j=\ell}^s \alpha_k = N - a_{\ell-1} \quad (2.14)$$

one gets

$$\sum_{j=1}^{\ell} \tau_j^{(-)} = a_\ell \left(-\frac{\pi - \zeta}{2\pi} + \frac{N(\pi - 2\zeta)}{2\pi L} \right) - a_\ell^2 \frac{(\pi - 2\zeta)}{2\pi L} - \sum_{k=1}^{a_\ell} \frac{n_{\sigma(k)}}{L} , \quad (2.15)$$

and also

$$\sum_{j=\ell}^s \tau_j^{(+)} = \tilde{a}_\ell \left(\frac{\pi - \zeta}{2\pi} - \frac{N(\pi - 2\zeta)}{2\pi L} \right) + \tilde{a}_\ell^2 \frac{N(\pi - 2\zeta)}{2\pi L} - \sum_{k=1}^{\tilde{a}_\ell} \frac{n_{\sigma(N-k+1)}}{L} . \quad (2.16)$$

Now by running through all the possible types of inequalities (2.13), one concludes that $S_\ell(\boldsymbol{\mu}) \rightarrow +\infty$ for any $\boldsymbol{\mu} \rightarrow \infty$, provided that, for any $\mathcal{J} \subset \llbracket 1 ; N \rrbracket$, the integers n_a satisfy

$$m \left(-\frac{\pi - \zeta}{2\pi} + \frac{N(\pi - 2\zeta)}{2\pi L} \right) - m^2 \frac{\pi - 2\zeta}{2\pi L} < \sum_{a \in \mathcal{J}} \frac{n_a}{L} < m \left(\frac{\pi - \zeta}{2\pi} - \frac{N(\pi - 2\zeta)}{2\pi L} \right) + m^2 \frac{\pi - 2\zeta}{2\pi L} \quad (2.17)$$

with $m = \#\mathcal{J}$.

Since, when (2.17) are satisfied, $S_\ell(\boldsymbol{\mu}) \rightarrow +\infty$ for any $\boldsymbol{\mu} \rightarrow \infty$, it follows that S_ℓ admits at least one minimum on \mathbb{R}^N at some point $\boldsymbol{\lambda}$. The coordinates of the point realising this minimum satisfy to the logarithmic Bethe equations, hence ensuring the existence of solutions.

It now remains to check that the constraints (2.17) are always satisfied provided that the bounds (2.2) hold. Let n_0 be such that

$$p_1 < \cdots < p_{n_0} \leq 0 < N < p_{n_0+1} < \cdots < p_n . \quad (2.18)$$

Then, for any $\mathcal{J} \subset \llbracket 1 ; N \rrbracket$, $\#\mathcal{J} = m$, agreeing upon $m_0 = \min(m, n_0)$, it holds

$$\sum_{a \in \mathcal{J}} n_a \geq \sum_{a=1}^{m_0} \left\{ p_a - \frac{N+1}{2} \right\} + \sum_{a=1}^{m-m_0} \left\{ a - \frac{N+1}{2} \right\} \geq m_0(p_1 - m + m_0 - 1) + \frac{m(m+1)}{2} - \frac{m(N+1)}{2} \quad (2.19)$$

where the lowest bound has been obtained by using that $p_a \geq p_1 + a - 1$. It thus follows that the lowest bound in (2.17) holds provided that

$$-m \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} + \frac{m}{L} \right) < \frac{m_0}{L} (p_1 - m + m_0 - 1) \quad (2.20)$$

Since $m_0 \leq m$ for $m \in \llbracket 1 ; n_0 \rrbracket$, these bounds are clearly satisfied as soon as (2.2) holds. Further, for $m > n_0$, the bounds (2.20) will hold provided that they hold for the smallest possible choice of p_1 compatible with (2.20), namely by taking it equal to the *lhs* of (2.20) when at $m = m_0 = 1$. This translates itself on the condition for n_0 :

$$-(m - n_0) \left[\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} + \frac{m + n_0}{L} \right) - \frac{n_0}{L} \right] - n_0(n_0 - 1) \frac{\pi - \zeta}{\pi L} \leq 0. \quad (2.21)$$

Since $m \geq n_0$, for the above to hold, it is enough that the latter always holds for $0 < \zeta \leq \pi/2$ and imposes the constraint

$$\frac{\pi - \zeta}{2\zeta - \pi} \left(\frac{1}{2} - \frac{N}{L} \right) \geq \frac{n_0}{L} \quad (2.22)$$

which is clearly satisfied provided that n is bounded according to (2.2). One can repeat the same reasoning relatively to the upper bound in (2.17) what leads to the sufficient constraints

$$-m \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} + \frac{m}{L} \right) > \frac{m_{\uparrow}}{L} (p_n - N + m - m_{\uparrow}) \quad \text{with} \quad m_{\uparrow} = \min(m, n - n_0). \quad (2.23)$$

For the latter to hold, it is enough to have

$$\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} + \frac{1}{L} \right) > \frac{m_{\uparrow}}{L} (p_n - N + m_{\uparrow} - m) \quad \text{and} \quad \frac{\pi - \zeta}{2\zeta - \pi} \left(\frac{1}{2} - \frac{N}{L} \right) \geq \frac{n}{L}. \quad (2.24)$$

Clearly, both bounds hold if (2.2) holds.

Now, regarding to uniqueness, observe that for $\zeta \in [\pi/2 ; \pi[$ $S_{\ell}(\boldsymbol{\mu})$ is strictly convex since it has a strictly positive defined Hessian matrix

$$\frac{\partial^2}{\partial \mu_k \partial \mu_j} \cdot S_{\ell}(\boldsymbol{\mu}) = \left[\frac{p'(\mu_k)}{2\pi} - \frac{1}{L} \sum_{b=1}^N K(\mu_k - \mu_b) \right] \delta_{jk} + \frac{K(\mu_j - \mu_k)}{L}. \quad (2.25)$$

Indeed, given $(h_1, \dots, h_N) \in \mathbb{R}^N$, one has that

$$\sum_{j,k=1}^N h_j h_k \frac{\partial^2}{\partial \mu_k \partial \mu_j} \cdot S_{\ell}(\boldsymbol{\mu}) = \sum_{k=1}^N h_k^2 \frac{p'(\mu_k)}{2\pi} - \sum_{j,k=1}^N \frac{(h_j - h_k)^2}{2} \cdot \frac{K(\mu_j - \mu_k)}{L} > 0. \quad (2.26)$$

Thus $S_{\ell}(\boldsymbol{\mu})$ admits a most a single minimum, in this range of ζ 's.

- $\Delta > 1$

The analysis follows basically the same lines as for $-1 < \Delta \leq 1$. Defining S_ℓ as in (2.4) with, now, the functions P and Θ being given by

$$P(\lambda) = \int_0^\lambda \mathfrak{p}(\mu) d\mu = \frac{|\lambda|^2}{2} + O(|\lambda|) \quad \text{and} \quad \Theta(\lambda) = \int_0^\lambda \theta(\mu) d\mu = \frac{|\lambda|^2}{2} + O(|\lambda|) \quad (2.27)$$

one gets that the logarithmic Bethe equations do arise as necessary conditions for a local extremum of S_ℓ . Sending μ to ∞ exactly as it was done for $-1 < \Delta \leq 1$ leads to the asymptotic behaviour

$$S_\ell(\mu) = \frac{t^2}{8\pi L} \sum_{a,b=1}^s \alpha_a \alpha_b \left\{ \left(\frac{L}{N} - 2 \right) (v_a^2 + v_b^2) + (v_a + v_b)^2 \right\} + O(t). \quad (2.28)$$

Since $L/N \geq 2$, this ensures that S_ℓ blows up at infinity and thus admits at least one minimum. ■

2.2 Boundedness of solutions to the logarithmic Bethe equations

I now prove that for $-1 < \Delta \leq 1$ at density $D \in [0; 1/2[$ and under certain restrictions on the allowed range for the p_a 's and n , the solutions to the logarithmic Bethe equations are bounded uniformly in N, L . Boundedness also holds for $\Delta > 0$ and $D \in [0; 1/2]$ under slightly different restrictions on the p_a 's. When $D = 1/2$ and $-1 < \Delta \leq 1$, the Bethe roots are not bounded any more. I establish bounds on the proportion of roots lying outside of a compact of large size.

Proposition 2.2.

- Let $-1 < \Delta \leq 1$

Let N, L be such that $N/L \rightarrow D \in [0; 1/2[$. Parametrise $\Delta = \cos(\zeta)$ with $\zeta \in [0; \pi[$ and let $\lambda_1, \dots, \lambda_N$ correspond to a solution of the logarithmic Bethe Ansatz equations (0.7) where the integers $p_1 < \dots < p_n$ are subject to the constraint

$$\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N-1}{L} \right) > \frac{p_n - N}{L}, \quad \frac{p_1 - 1}{L} > -\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N-1}{L} \right) \quad (2.29)$$

and n is fixed and N independent. Then, there exists a N, L independent constants $\Lambda > 0$ such that

$$|\lambda_a| \leq \Lambda \quad \text{for any } a \in \llbracket 1; N \rrbracket \text{ such that } \ell_a \in \llbracket 1; N \rrbracket \quad \text{uniformly in } (N, L). \quad (2.30)$$

- Let $\Delta > 1$.

Let N, L be such that $N/L \rightarrow D \in [0; 1/2]$ and let $\lambda_1, \dots, \lambda_N$ correspond to the solution of the logarithmic Bethe Ansatz equations (0.7) where the integers $p_1 < \dots < p_n$ are such that n is fixed, L independent, and $|p_a/L| \leq C$ for some L -independent $C > 0$.

Then, there exists a N, L independent constants $\Lambda > 0$ such that

$$|\lambda_a| \leq \Lambda \quad \text{for any } a \in \{1, \dots, N\} \quad \text{uniformly in } (N, L). \quad (2.31)$$

Note that given the constraints on the p_a 's, solutions to the logarithmic Bethe Ansatz equations do exist in virtue of Proposition 2.1.

Proof —

The proof splits in three steps since one has to distinguish between $-1 < \Delta \leq 0$, $0 < \Delta \leq 1$ and $\Delta > 1$. The distinction between the first two regimes is due to the change in the sign of the kernel $K(\lambda)$. The last regime has to be treated separately due to the change in the periodicity properties of the involved functions.

- $\Delta > 1$

Let $\{\check{\lambda}_a\}_1^N$ be the reordered λ_a 's, namely $\check{\lambda}_a = \lambda_{\ell_a}$ and $\check{\lambda}_1 < \dots < \check{\lambda}_N$. Also, let $\{\check{\ell}_a\}$ denote the corresponding reordering of the ℓ_a 's. Let

$$\widehat{\xi}(\lambda) = \frac{1}{2\pi} \mathfrak{p}(\lambda) - \frac{1}{2\pi L} \sum_{a=1}^N \theta(\lambda - \lambda_a) + \frac{N+1}{2L} \quad (2.32)$$

be the counting function built up from this solution to the Bethe Ansatz equations.

Since the λ_a 's are real, by π -periodicity of the functions, it follows that

$$|\widehat{\xi}'(\lambda)| \leq \frac{1}{2\pi} \mathfrak{p}'(0) + \frac{1}{2} K(0). \quad (2.33)$$

Let p be such that $(p+1)\pi > \check{\lambda}_N - \check{\lambda}_1 \geq p\pi$. Then, since $\widehat{\xi}(x+k\pi) - \widehat{\xi}(x) = k(L-N)/L$, one has

$$\frac{\check{\ell}_N - \check{\ell}_1}{L} = \widehat{\xi}(\check{\lambda}_N) - \widehat{\xi}(\check{\lambda}_1) = \widehat{\xi}(\check{\lambda}_1 + p\pi) - \widehat{\xi}(\check{\lambda}_1) + \int_{\check{\lambda}_1 + p\pi}^{\check{\lambda}_N} \widehat{\xi}'(\lambda) \cdot d\lambda \geq p \frac{L-N}{L} - \frac{1}{2} \mathfrak{p}'(0) - \frac{\pi}{2} K(0). \quad (2.34)$$

This yields the upper bound on p since the $\check{\ell}_a$'s are bounded in L . Thus there exists an L -independent constant $\widetilde{C} > 0$ such that $\widetilde{C} > |\check{\lambda}_N - \check{\lambda}_1| > 0$. Since θ is increasing on \mathbb{R} , it holds

$$\frac{1}{2\pi} \mathfrak{p}(\check{\lambda}_N) \leq \frac{1}{L} \max\{\ell_a\} - \frac{N+1}{2L} + \frac{N}{2\pi L} \theta(\widetilde{C}) = \frac{C'}{2\pi} \Rightarrow \check{\lambda}_N \leq (\mathfrak{p})^{-1}(C') = \Lambda \quad (2.35)$$

The lower bound on $\check{\lambda}_1$ is obtained analogously.

- $\Delta = \cos(\zeta)$ with $\zeta \in [0; \pi/2]$

Reorganise the Bethe roots as

$$\{\lambda_a\}_1^N = \{\nu_a\}_1^N \quad \text{with} \quad \nu_a = \lambda_{q_a} \quad \text{and} \quad \nu_{n+1} < \dots < \nu_N \quad \text{with} \quad \ell_{q_a} \in \llbracket 1; N \rrbracket \quad \text{for} \quad a = n+1, \dots, N. \quad (2.36)$$

Since θ is increasing in this region of ζ 's and is bounded by $\pi - 2\zeta$, one has

$$\frac{\mathfrak{p}(\nu_N)}{2\pi} < \frac{1}{L}(\ell_{q_N} - N) + \frac{N-1}{2L} + \frac{\pi-2\zeta}{2\pi L} N. \quad (2.37)$$

Since $\ell_{q_N} \leq N$

$$\frac{\mathfrak{p}(\nu_N)}{2\pi} < \frac{\pi-\zeta}{\pi} \cdot \frac{N}{L} < \frac{\pi-\zeta}{\pi} (D + \epsilon) \Leftrightarrow \check{\lambda}_N \leq (\mathfrak{p})^{-1}(2(\pi-\zeta)(D + \epsilon)). \quad (2.38)$$

where $\epsilon > 0$ is such that $N/L < D + \epsilon < 1/2$ and L is taken large enough.

The bound on $\check{\lambda}_1$ is obtained analogously, leading to

$$\frac{\mathfrak{p}(\nu_{n+1})}{2\pi} > -\frac{\pi-\zeta}{\pi} (D + \epsilon) \Leftrightarrow \nu_{n+1} \geq -\mathfrak{p}^{-1}(2(\pi-\zeta)(D + \epsilon)). \quad (2.39)$$

‡. There cannot be two λ_a 's that are equal since this would contradict that the ℓ_a 's are pairwise distinct.

Thus, the claim holds with $\Lambda = p^{-1}(2(\pi - \zeta)(D + \epsilon))$.

- $\Delta = \cos(\zeta)$ with $\zeta \in [\pi/2; \pi[$

It is a classical fact that if $\{\lambda_a\}_1^N$ solve the system of logarithmic Bethe equations associated with the choice of integers ℓ_a , then the set of roots $\{\mu_a\}_1^N$ with $\mu_a = -\lambda_{N+1-a}$ solve the system of logarithmic Bethe equations associated with the choice of integers $-\ell_{N+1-a}$. Therefore, the claim will follow as soon as one shows that $\limsup_{N,L \rightarrow \infty} [\nu_N] < +\infty$, where the ν_a 's are as defined in (2.36).

The proof goes by contradiction. Hence, assume that $\limsup_{N,L \rightarrow \infty} [\nu_N] = +\infty$. In other words, there exists a sequence (L_m, N_m) , $N_m, L_m \rightarrow +\infty$, $N_m/L_m \rightarrow D$ and the associated Bethe roots $\{\lambda_a^{(m)}\}_1^{N_m}$ associated with the subsequences $\{\ell_a^{(m)}\}_1^{N_m}$ and chains of L_m sites such that $\lim_{m \rightarrow +\infty} \nu_N^{(m)} = +\infty$. Fix $Q > 0$ and set

$$\mathfrak{b}_Q^{(m)} = \frac{1}{N_m - n} \# \{a \in \llbracket n+1; N_m \rrbracket : \nu_a^{(m)} \geq Q\}. \quad (2.40)$$

Define further

$$\bar{\mathfrak{b}}_Q = \limsup_{m \rightarrow +\infty} \mathfrak{b}_Q^{(m)} \quad \text{and} \quad \bar{\mathfrak{b}} = \limsup_{Q \rightarrow +\infty} \bar{\mathfrak{b}}_Q. \quad (2.41)$$

By construction, one has $1 \geq \bar{\mathfrak{b}} \geq 0$. Furthermore, by definition, there exists an increasing sequence Q_ℓ , $Q_\ell \xrightarrow{\ell \rightarrow +\infty} +\infty$, such that $\bar{\mathfrak{b}}_{Q_\ell} \rightarrow \bar{\mathfrak{b}}$.

Assume that $1 \geq \bar{\mathfrak{b}} > 0$ and pick any $\epsilon > 0$ such that $\bar{\mathfrak{b}} - \epsilon > 0$. Let ℓ_0 be such that

$$\forall \ell \geq \ell_0 \quad \bar{\mathfrak{b}} - \frac{\epsilon}{2} < \bar{\mathfrak{b}}_{Q_\ell} < \bar{\mathfrak{b}} + \frac{\epsilon}{2}. \quad (2.42)$$

Also, given a fixed $\ell \geq \ell_0$, there exists a sequence $r \mapsto m_r(\ell)$ and $r_0(\ell) \in \mathbb{N}$ such that for any $r \geq r_0(\ell)$ one has

$$\bar{\mathfrak{b}} - \epsilon < \bar{\mathfrak{b}}_{Q_\ell} - \frac{\epsilon}{2} < \bar{\mathfrak{b}}_{Q_\ell}^{(m_r(\ell))} < \bar{\mathfrak{b}}_{Q_\ell} + \frac{\epsilon}{2} < \bar{\mathfrak{b}} + \epsilon. \quad (2.43)$$

One can slightly improve the *rhs* of the bound. Indeed, since $\bar{\mathfrak{b}}_{Q_{\ell_0}} < \bar{\mathfrak{b}} + \epsilon/2$, there exists m_0 such that for any $m \geq m_0$, one has $\mathfrak{b}_{Q_{\ell_0}}^{(m)} < \bar{\mathfrak{b}} + \epsilon$. As $m_r(\ell)$ goes to infinity with r , there exists

$$r'_1(\ell) \quad \text{such that} \quad \forall r \geq r'_1(\ell) \quad \text{one has} \quad m_r(\ell) \geq m_0. \quad (2.44)$$

Thence, setting $r_1(\ell) = \max(r_0(\ell), r'_1(\ell))$, one has for any $r \geq r_1(\ell)$ that

$$(\bar{\mathfrak{b}} + \epsilon)(N_{m_r(\ell)} - n) \geq \# \{a \in \llbracket 1+n; N_{m_r(\ell)} \rrbracket : \nu_a^{(m_r(\ell))} \geq Q_{\ell_0}\}. \quad (2.45)$$

Since, the total number of the λ 's is $N_{m_r(\ell)}$, the above inequality implies that

$$\# \{a \in \llbracket 1+n; N_{m_r(\ell)} \rrbracket : \nu_a^{(m_r(\ell))} < Q_{\ell_0}\} \geq N_{m_r(\ell)} \cdot \max\{0, 1 - \epsilon - \bar{\mathfrak{b}}\}. \quad (2.46)$$

$$\bar{a} = [N_{m_r(\ell)}(1 + \epsilon - \bar{\mathfrak{b}})] + 1 \quad \text{and} \quad \underline{a} = \max\{0, [N_{m_r(\ell)}(1 - \epsilon - \bar{\mathfrak{b}})]\} \quad (2.47)$$

where $[.]$ stands for the integer part. Then,

$$\sum_{k=\bar{a}}^{N_{m_r(\ell)}} \ell_{q_k} \leq \sum_{k=\bar{a}+n}^{N_{m_r(\ell)}} k \leq \frac{1}{2}(N_{m_r(\ell)} - \bar{a} + 1)(N_{m_r(\ell)} - \bar{a} - n + 1). \quad (2.48)$$

Therefore, after summing up the logarithmic Bethe equations involving $v_{\bar{a}}, \dots, v_{N_{m_r(\ell)}}$, invoking the above bounds and using that θ is odd, one gets

$$(N_{m_r(\ell)} - \bar{a} + 1) \frac{\bar{a} - 1}{2L_{m_r(\ell)}} \geq \frac{1}{2\pi} \sum_{k=\bar{a}}^{N_{m_r(\ell)}} \mathfrak{p}(v_k^{(m_r(\ell))}) - \frac{1}{2\pi L_{m_r(\ell)}} \sum_{k=\bar{a}}^{N_{m_r(\ell)}} \sum_{p=\underline{a}+1}^{\bar{a}-1} \theta(v_k^{(m_r(\ell))} - v_p^{(m_r(\ell))}) \\ - \frac{1}{2\pi L_{m_r(\ell)}} \sum_{k=\bar{a}}^{N_{m_r(\ell)}} \sum_{p=1}^{\underline{a}} \theta(v_k^{(m_r(\ell))} - v_p^{(m_r(\ell))}) - \frac{1}{2\pi L_{m_r(\ell)}} \sum_{k=\bar{a}}^{N_{m_r(\ell)}} \sum_{p=1}^n \theta(v_k^{(m_r(\ell))} - v_p^{(m_r(\ell))}). \quad (2.49)$$

Using that $-\theta(\lambda)$ is strictly increasing so that for $k > p > n$ one has $-\theta(v_k^{(m_r(\ell))} - v_p^{(m_r(\ell))}) > -\theta(0) = 0$ and that by (2.43)

$$\check{\lambda}_k > Q_\ell \quad \text{for } k \in \llbracket \bar{a}; N_{m_r(\ell)} \rrbracket \quad \text{and} \quad \check{\lambda}_k < Q_{\ell_0} \quad \text{for } k \in \llbracket 1; \underline{a} \rrbracket \quad (2.50)$$

by (2.45), one arrives to the bound

$$\frac{\bar{a} - 1}{2L_{m_r(\ell)}} \geq \frac{\mathfrak{p}(Q_\ell)}{2\pi} - \frac{\underline{a}}{2\pi L_{m_r(\ell)}} \theta(Q_\ell - Q_{\ell_0}) - \frac{2\zeta - \pi}{2\pi L_{m_r(\ell)}} n. \quad (2.51)$$

Taking the $r \rightarrow +\infty$ limit, it follows from

$$\lim_{r \rightarrow +\infty} (\bar{a}/L_{m_r(\ell)}) = D \cdot (1 + \epsilon - \bar{b}) \quad \text{and} \quad \lim_{r \rightarrow +\infty} (\underline{a}/L_{m_r(\ell)}) = D \cdot \max\{0, (1 - \epsilon - \bar{b})\}, \quad (2.52)$$

that

$$\frac{D}{2} (1 + \epsilon - \bar{b}) \geq \frac{\mathfrak{p}(Q_\ell)}{2\pi} - \frac{D}{2\pi} \max\{0, (1 - \epsilon - \bar{b})\} \vartheta(Q_\ell - Q_{\ell_0}). \quad (2.53)$$

Sending first $\ell \rightarrow +\infty$ and then $\epsilon \rightarrow 0^+$ leads to

$$D \frac{1 - \bar{b}}{2} \geq \frac{\pi - \zeta}{2\pi} - D \frac{\pi - 2\zeta}{2\pi} (1 - \bar{b}) \quad \text{viz.} \quad D \cdot (1 - \bar{b}) \geq \frac{1}{2}. \quad (2.54)$$

However, the last inequality cannot hold since, by hypothesis, $0 < D < 1/2$.

Thus, one necessarily has $\bar{b} = 0$. This however does not yet guarantee that the roots are bounded from above since a small portion thereof can escape to $+\infty$. Thus assume that one has $\check{\lambda}_{N_{m_r(\ell)}}^{(m_r(\ell))} \rightarrow +\infty$. Then, starting from the logarithmic Bethe equations for $\check{\lambda}_{N_{m_r(\ell)}}^{(m_r(\ell))}$, and using similar bounds one gets

$$\frac{1}{L_{m_r(\ell)}} \left(\frac{N_{m_r(\ell)} - 1}{2} \right) \geq \frac{\mathfrak{p}(v_{N_{m_r(\ell)}})}{2\pi} - \frac{\underline{a}}{2\pi L_{m_r(\ell)}} \theta(v_{N_{m_r(\ell)}} - Q_{\ell_0}) - \frac{n(2\zeta - \pi)}{2\pi L_{m_r(\ell)}} \quad (2.55)$$

with \underline{a} and other quantities as defined above with the difference that, now, $\bar{b} = 0$. One can send $r \rightarrow +\infty$ on the level of these bounds and then relax $\epsilon \rightarrow 0$ leading to (2.54) with $\bar{b} = 0$. This yields the sought contradiction. ■

In the rest of this subsection, I focus on the case $-1 < \Delta \leq 1$ when $D = 1/2$ and obtain bounds on the proportion of Bethe roots for the ground state which lie away from some segment I_Λ . More precisely, one has the

Proposition 2.3. *Let $-1 < \Delta \leq 1$ and $N/L = D = 1/2$. Let $\{\lambda_a\}_1^N$ be a solution to the logarithmic Bethe Ansatz equations associated with the ground state choice of integers $\ell_a = a$, with $a \in \llbracket 1; N \rrbracket$. Then, for any $\epsilon > 0$ there exists $\Lambda_\epsilon, L_0 > 0$, such that for any $L \geq L_0$,*

$$\widehat{c}_{\Lambda_\epsilon} \leq \epsilon \quad \text{where} \quad \widehat{c}_\Lambda = \frac{1}{L} \# \{a \in \llbracket 1; N \rrbracket : \lambda_a \in \mathbb{R} \setminus I_\Lambda\} \quad (2.56)$$

corresponds to the fraction of Bethe roots lying away from the segment I_Λ .

Proof —

- $0 \leq \Delta \leq 1$

Straightforward bounds analogous to those developed in the proof of Proposition 2.2 lead to the upper bounds

$$\frac{N-a}{L} + \frac{1}{2L} < \int_{\lambda_a}^{+\infty} p'(\mu) \cdot \frac{d\mu}{2\pi} \quad \text{and} \quad \frac{a}{L} < \frac{1}{2L} \int_{-\infty}^{\lambda_a} p'(\mu) \cdot \frac{d\mu}{2\pi}. \quad (2.57)$$

Assume that $\lambda_a > \Lambda$. Then, one gets the upper bound

$$\frac{N-a}{L} < \int_{\Lambda}^{+\infty} p'(\mu) \cdot \frac{d\mu}{2\pi} \quad \text{and setting} \quad \widehat{k}_{\Lambda}^+ = \left\lfloor L \int_{\Lambda}^{+\infty} p'(\mu) \cdot \frac{d\mu}{2\pi} \right\rfloor + 1 \quad (2.58)$$

where $[*]$ denotes the integer part, it is clear that there can be at most \widehat{k}_{Λ}^+ distinct integers $\ell_a \in \llbracket 1; N \rrbracket$ satisfying to this constraint. Likewise, if $\lambda_a < -\Lambda$, one has

$$\frac{a}{L} < \int_{-\infty}^{-\Lambda} p'(\mu) \cdot \frac{d\mu}{2\pi} + \frac{1}{2L}. \quad \text{Thus setting} \quad \widehat{k}_{\Lambda}^- = \left\lfloor L \int_{-\infty}^{-\Lambda} p'(\mu) \cdot \frac{d\mu}{2\pi} \right\rfloor + 2 \quad (2.59)$$

it becomes clear that there can be at most \widehat{k}_{Λ}^- distinct integers $\ell_a \in \llbracket 1; N \rrbracket$ satisfying to the above constraint. Hence, all-in-all, one gets the bound

$$\widehat{c}_{\Lambda} \leq \frac{3}{L} \int_{\mathbb{R} \setminus I_{\Lambda}} p'(\mu) \cdot \frac{d\mu}{2\pi}. \quad (2.60)$$

The conclusion then follows from the fact that the *rhs* of the inequality approaches 0 when $L, \Lambda \rightarrow +\infty$.

- $-1 < \Delta < 0$

If the sequence $\{\lambda_a\}_1^N$ is bounded uniformly in N , then the statement holds simply by taking $\Lambda_{\epsilon} > M$, where M is a bound on the magnitude of the Bethe roots. Else, one reasons as in the proof of Proposition 2.2 so as to establish that $\bar{b} = 0$ with \bar{b} as defined in (2.41). One gets similar bounds relatively to the proportion of roots lower than some fixed parameter Q . The rest is straightforward. ■

3 Asymptotic expansion of the counting function

3.1 Leading asymptotics of the counting function

Given a solution $\{\lambda_a\}_1^N$ to the logarithmic Bethe equations associated with the choice of integers $\{\ell_a\}_1^N$, it is convenient to introduce the associated counting function

$$\widehat{\xi}(\lambda) = \frac{1}{2\pi} p(\lambda) - \frac{1}{2\pi L} \sum_{a=1}^N \theta(\lambda - \lambda_a) + \frac{N+1}{2L}. \quad (3.1)$$

By construction, this function is such that $\widehat{\xi}(\lambda_a) = \ell_a$ for $a = 1, \dots, N$. The purpose of this section is to provide an alternative to (3.1) characterisation of the counting function valid in the large- L regime. This characterisation

is obtained by means of a non-linear integral equation. The very idea goes back to the works of De Vega, Woynarowich [9] and Batchelor, Klümper [22] and was further developed in the works [7, 23]. The non-linear integral equations obtained in the earlier literature were obtained by *assuming* that the counting function satisfies certain properties such as being strictly increasing on the real axis and, on compact subsets of \mathbb{R} , having a bounded from below derivative, $\widehat{\xi}' > \kappa > 0$, this uniformly in L . The main point of the method is that once a non-linear integral equation is taken for granted just as certain amount of properties of its solution, then it is relatively easy to compute, order-by-order, the coefficients of its large- L asymptotic expansion. The assumptions which allow one to derive the non-linear integral equation for the counting function and which also allow one to derive its large- L asymptotic expansion could, in the best case scenario, be verified *a posteriori*, namely on the level of the obtained form for the large- L asymptotic expansion. This only allowed for a consistency test of the calculations.

The main input of the analysis that I develop below is to set techniques allowing one

- i) to prove that, for L -large enough, the counting function can indeed be characterised as the unique solution to a non-linear integral and that it does indeed satisfy to the expected properties, in particular, that it is strictly increasing on \mathbb{R} ;

ii) to demonstrate that the counting function admits a large- L asymptotic expansion up to $o(L^{-1})$ corrections. In other words, the framework developed below allows one to step out of the formal handling of asymptotic expansions. Prior to stating the result and going into the details of the proof, I need to introduce several building blocks of the non-linear integral equations satisfied by $\widehat{\xi}$.

Given $\widehat{D} = N/L \in [0; 1/2]$, here and in the following \widehat{q} will denote the unique solution to the magnetic Fermi boundary problem (1.25) associated with \widehat{D} . In its turn, q will denote its thermodynamic limit, *viz.* the unique solution to the Fermi boundary problem (1.25) associated with $D = \lim \widehat{D}$.

Definition 3.1. *Let f be real analytic function such that it that is a biholomorphism on some open neighbourhood of a segment of \mathbb{R} and such that its range contains, for some $\alpha_0 > 0$ small enough, the closed set $\widehat{G}(\alpha_0)$ depicted on Fig. 1. Then, one can construct the three below non-linear integral operators*

$$\mathfrak{R}_{N;1}[f](\lambda) = - \sum_{\epsilon=\pm} \int_{\widehat{\Gamma}_\epsilon} \frac{R_{I_{\widehat{q}}}(\lambda, f^{-1}(z))}{f'(f^{-1}(z))} \cdot \ln[1 - e^{2i\pi\epsilon Lz}] \cdot \frac{dz}{2i\pi L} \quad (3.2)$$

and agreeing upon

$$\widehat{q}_R[f] = f^{-1}\left(\frac{N+1/2}{L}\right) \quad , \quad \widehat{q}_L[f] = f^{-1}\left(\frac{1}{2L}\right) \quad (3.3)$$

one has

$$\mathfrak{R}_{N;2}[f](\lambda) = - \int_{\widehat{q}}^{\widehat{q}_R[f]} R_{I_{\widehat{q}}}(\lambda, \mu) [f(\mu) - f(\widehat{q}_R[f])] \cdot d\mu \quad \text{and} \quad \mathfrak{R}_{N;3}[f](\lambda) = - \int_{\widehat{q}_L[f]}^{-\widehat{q}} R_{I_{\widehat{q}}}(\lambda, \mu) [f(\mu) - f(\widehat{q}_L[f])] \cdot d\mu \quad (3.4)$$

These operators, build up a "master" non-linear integral operator as

$$\mathfrak{R}_N[f](\lambda) = \sum_{a=1}^3 \mathfrak{R}_{N;a}[f](\lambda) \quad (3.5)$$

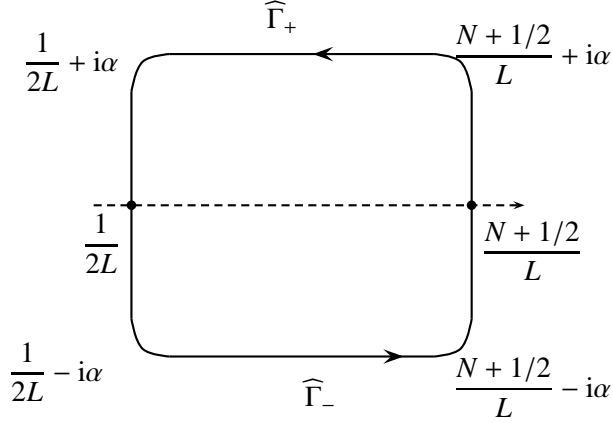


FIGURE 1 – Closed set $\widehat{G}(\alpha)$ delimited by its boundary $\widehat{\Gamma}_+ \cup \widehat{\Gamma}_-$.

Just as in Section 2, I will consider

$$h_1 < \dots < h_n, h_a \in \llbracket 1; N \rrbracket \quad \text{and} \quad p_1 < \dots < p_n, p_a \in \mathbb{Z} \setminus \llbracket 1; N \rrbracket \quad (3.6)$$

where $n \in \mathbb{N}$ is N -independent. However, the integers h_a and p_a can depend on N provided that they satisfy to certain bounds :

- for $-1 < \Delta \leq 1$, $\Delta = \cos(\zeta)$, in addition, the integers $\{p_a\}_1^n$ are assumed to satisfy to the additional constraint

$$\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N-1}{L} \right) > \frac{p_n - N}{L} \quad , \quad \frac{p_1 - 1}{L} > -\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N-1}{L} \right). \quad (3.7)$$

- For $\Delta > 1$, the integers $\{p_a\}_1^n$ are solely assumed to be bounded as

$$|p_a/L| \leq C \quad \text{for some } L\text{-independent } C > 0. \quad (3.8)$$

In the course of the proof, it will be useful to introduce the sets

$$\mathcal{S}_\eta(J) = \{z \in \mathbb{C} : |\Im(z)| < \eta, \Re(z) \in J\} \quad (3.9)$$

and

$$\mathcal{S}_{\eta,\epsilon}(J) = \{z \in \mathbb{C} : |\Im(z)| < \eta, d(\Re(z), J) < \epsilon\} \quad (3.10)$$

where $J \subset \mathbb{R}$ is a subset of \mathbb{R} and d is the distance between subsets of \mathbb{R} induced by the Euclidian distance.

Finally, it will also appear convenient to introduce the parameter \varkappa_Δ defined by

$$\begin{array}{lll} -1 < \Delta < 1 & \varkappa_\Delta = \zeta/4 & \Delta = \cos(\zeta) \quad 0 < \zeta < \pi \\ \Delta = 1 & \varkappa_\Delta = 1/4 & \\ \Delta > 1 & \varkappa_\Delta = \zeta/4 & \Delta = \cosh(\zeta) \quad \zeta > 0 \end{array} \quad (3.11)$$

Theorem 3.2. Given $\Delta > -1$, let $n \in \mathbb{N}$ be fixed and the integers $\{\ell_a\}_1^N$ be defined in terms of integers $\{p_a\}_1^n$ and $\{h_a\}_1^n$ according to (2.3). Assume that these integers satisfy, depending on the value of Δ , the conditions (3.7)-(3.8). Let N, L be such that $0 \leq \widehat{D} = N/L \leq 1/2$ and go to infinity so that $\widehat{D} \rightarrow D$ with

$$D \in [0; 1/2[\quad \text{for } -1 < \Delta \leq 1 \quad \text{and} \quad D \in [0; 1/2] \quad \text{for } \Delta > 1. \quad (3.12)$$

Finally, let $\{\lambda_a\}_1^N$ denote a solution to the logarithmic Bethe equations subordinate to the associated integers $\{\ell_a\}_1^N$. Then

i) there exists L_0 large enough such that, for any $L \geq L_0$ the associated counting function (3.1) is a strictly increasing diffeomorphism from

$$\mathbb{R} \quad \text{onto} \quad \left[-\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} \right) + \frac{1}{2L}; \frac{N}{L} + \frac{1}{2L} + \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N}{L} \right) \right] \quad (3.13)$$

when $-1 < \Delta \leq 1$ while, for $\Delta > 1$, $\widehat{\xi}$ is a diffeomorphism from \mathbb{R} onto \mathbb{R} .

ii) There exists an open neighbourhood \widehat{U} of $[-q; q]$ and an L -independent open neighbourhood V_D of $[0; D]$ containing $\widehat{G}(\alpha)$ delimited by the curves $\widehat{\Gamma}_\pm$, c.f. Fig. 1, for some $\alpha_0 > 0$ such that $\widehat{\xi} : \widehat{U} \mapsto V_D$ is a biholomorphism.

iii) There exists $C > 0$ such that

$$\|\widehat{\xi}(\cdot) - \xi_0(\cdot | \widehat{q})\|_{L^\infty(\mathcal{S}_{\Delta}(\mathbb{R}))} \leq \frac{C}{L} \quad \text{with} \quad \xi_0(\lambda | \widehat{q}) = p(\lambda | \widehat{q}) + \frac{\widehat{D}}{2}. \quad (3.14)$$

Above, $*$ denotes the running variable of the functions.

iv) The counting function solves the non-linear integral equation

$$\widehat{\xi}(\lambda) = \xi_0(\lambda | \widehat{q}) + \frac{1}{L} \cdot \Phi_q^{(0)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) + \Re_N[\widehat{\xi}](\lambda), \quad (3.15)$$

with

$$\widehat{x}_a = \widehat{\xi}^{-1}\left(\frac{a}{L}\right) \quad a \in \{h_1, \dots, h_n, p_1, \dots, p_n\}. \quad (3.16)$$

The function $\Phi_q^{(s)}$ is expressed in terms of the dressed phase and charge as

$$\Phi_q^{(s)}(\lambda | \{y_a\}_1^m; \{z_a\}_1^n) = \frac{1}{2} [1 + sZ(\lambda | \widehat{q})] - \sum_{a=1}^m \varphi(\lambda, y_a | \widehat{q}) + \sum_{a=1}^n \varphi(\lambda, z_a | \widehat{q}) \quad (3.17)$$

and the operator \Re_N is as given in Definition 3.1.

v) The non-linear integral equation (3.15) admits a unique solution in the class of functions satisfying to ii) and iii) when $\Delta > 1$ or $-1 < \Delta \leq 1$ and, on top of the previous conditions, the p_a/L all belong to an L -independent compact subset of

$$\left[-\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - D \right); D + \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - D \right) \right]. \quad (3.18)$$

The constant C in (3.14) and L_0 appearing above only depend on n and are uniform in \widehat{D} belonging to compact subsets of $[0; 1/2[$ for $-1 < \Delta \leq 1$ and throughout the segment $[0; 1/2]$ for $\Delta > 1$.

In virtue of Proposition 2.1, given integers $\{h_a\}_1^n$ and $\{p_a\}_1^n$ satisfying to the constraints (3.7)-(3.8), depending on the value of Δ , there are always solutions to the logarithmic Bethe Ansatz equations. When $-1 < \Delta \leq 0$, the solution was shown to be unique but, *a priori*, when $\Delta > 0$, there could exist more than one solution to the logarithmic Bethe Ansatz equations associated with this given choice of integers. The statement of the theorem does hold for any such solution.

Proof —

For each value of N, L , one is provided with integers $\{h_a\}_1^n$ and $\{p_a\}_1^n$, which, as mentioned, in most cases do depend on N and L . In virtue of Proposition 2.1, these give rise to a sequence $\{\lambda_a\}_1^N$, indexed by L , of solutions to the logarithmic Bethe Ansatz equations. By Proposition 2.2, equation (2.31), there exists and L -independent $\Lambda > 0$ such that :

- $|\lambda_a| \leq \Lambda$ for $a \in \llbracket 1 ; N \rrbracket$ if $\Delta > 1$,
- $|\lambda_a| \leq \Lambda$ for $a \in \llbracket 1 ; N \rrbracket$ such that $L\widehat{\xi}(\lambda_a) \in \llbracket 1 ; N \rrbracket$ in the regime $-1 < \Delta \leq 1$,

this uniformly in N, L . This sequence of Bethe roots $\{\lambda_a\}_1^N$ defines the associated counting function through (3.1), therefore giving rise to a sequence (in respect to L) of counting functions $\widehat{\xi}$. These counting functions are readily seen to be holomorphic functions on the strip $\mathcal{S}_{2\kappa_\Delta}(\mathbb{R})$. Furthermore, straightforward bounds show there exists an L -independent constant $B > 0$ such that

$$\|\widehat{\xi}\|_{L^\infty(\mathcal{S}_{2\kappa_\Delta}(I_{2\Lambda}))} \leq B. \quad (3.19)$$

Since $\widehat{\xi}$ is a sequence of holomorphic functions on $\mathcal{S}_{\kappa_\Delta}(I_{2\Lambda})$ that are uniformly bounded in L , by Montel's theorem, it admits a converging subsequence $\widehat{\xi}_e$ which converges, in the sup norm topology on compacts subsets, to a holomorphic function ξ_e on $\mathcal{S}_{\kappa_\Delta}(I_{2\Lambda})$.

The strategy of the proof consists in characterising the limit ξ_e of such a convergent subsequence. It will be shown that ξ_e necessarily coincides with the so-called thermodynamic counting function $\xi_0(\cdot | q)$, *c.f.* (3.14). Worded differently, any converging subsequence of $\widehat{\xi}$ has the *same* limit. Since, by Montel's theorem, any subsequence of $\widehat{\xi}$ admits a converging subsequence, $\widehat{\xi}$ necessarily converges to $\xi_0(\cdot | q)$. Once that the convergence is established, the form of the non-linear integral equation (3.15) satisfied by $\widehat{\xi}$ follows rather easily from the properties of the limit ξ_0 . A straightforward investigation of the non-linear integral equation ensures the uniqueness of solutions for L large-enough and the bounds (3.14).

In order to lighten the notations, I will subsequently drop the subscript e in all the considerations namely the subsequence and its limit will still be denoted by $\widehat{\xi}$ and ξ . Since ξ' is holomorphic on $I_{2\Lambda}$, it admits, a finite number of zeroes and thus may only change signs a finite number of times.

In the first part of the proof, I will assume that $\xi' > 0$ and show that this property is enough so as to characterise the limit. In the second part of the proof, I will establish the uniqueness of solutions to the non-linear integral equation. Finally, in the third part of the proof, I will rule out the possibility that ξ' has zeroes or changes sign on $[-\Lambda ; \Lambda]$.

- $\xi' > 0$ on $[-\Lambda ; \Lambda]$

In virtue of Proposition A.1, there exists $\eta, \epsilon > 0$ such that

$$\xi : \mathcal{S}_{2\eta, 2\epsilon}(I_\Lambda) \rightarrow \xi(\mathcal{S}_{2\eta, 2\epsilon}(I_\Lambda)) \text{ is a biholomorphism satisfying } \xi(\mathcal{S}_{2\eta, 2\epsilon}(I_\Lambda) \cap \mathbb{H}^\pm) = \xi(\mathcal{S}_{2\eta, 2\epsilon}(I_\Lambda)) \cap \mathbb{H}^\pm.$$

Furthermore, for L large enough, it holds that $\widehat{\xi}(\mathcal{S}_{2\eta, 2\epsilon}(I_\Lambda)) \supset \xi(\mathcal{S}_{\eta, \epsilon}(I_\Lambda)) \supset \widehat{\xi}(I_\Lambda)$ and

$$\widehat{\xi} : \mathcal{S}_{2\eta, 2\epsilon}(I_\Lambda) \cap \widehat{\xi}^{-1}(\xi(\mathcal{S}_{\eta, \epsilon}(I_\Lambda))) \rightarrow \xi(\mathcal{S}_{\eta, \epsilon}(I_\Lambda)) \quad (3.20)$$

is a biholomorphism. Besides $\widehat{\xi}$ is strictly increasing on $I_{\Lambda+\epsilon}$.

Since all the Bethe roots λ_a such that $L\widehat{\xi}(\lambda_a) \in \llbracket 1; N \rrbracket$ are contained in I_Λ where $\widehat{\xi}$ is increasing and owing to $\min\{\ell_a\} \leq n$ and $\max\{\ell_a\} \geq N - n$, it follows that

$$\left[\frac{n}{L}; \frac{N-n}{L} \right] \subset \widehat{\xi}(I_\Lambda). \quad (3.21)$$

Moreover, for L large enough

$$\widehat{\xi}(I_\Lambda) \subset \left\{ x \in \mathbb{R} : d(x, \xi(I_\Lambda)) < \|\widehat{\xi} - \xi\|_{L^\infty(\mathcal{S}_{\Delta}(I_{2\Lambda}))} \right\} \subset \xi(I_{\Lambda+\epsilon/2}). \quad (3.22)$$

Then, passing to the limit in (3.20), it holds that

$$[0; D] \subset \xi(I_{\Lambda+\epsilon/2}) \implies [-\nu; D + \nu] \subset \xi(I_{\Lambda+\epsilon}) \quad (3.23)$$

for some $\nu > 0$ small enough. This inclusion ensures that

$$\left[\frac{1}{2L}; \frac{N+1/2}{L} \right] \subset \xi(\mathcal{S}_{\eta,\epsilon}(I_\Lambda)). \quad (3.24)$$

Thus, due to (3.20), the points

$$\widehat{q}_L = \widehat{\xi}^{-1}\left(\frac{1}{2L}\right) \quad \text{and} \quad \widehat{q}_R = \widehat{\xi}^{-1}\left(\frac{N+1/2}{L}\right) \quad (3.25)$$

are well defined and belong to $\mathcal{S}_{\eta,\epsilon}(I_\Lambda)$. Moreover, when $\Delta > 1$, one has the bound

$$\widehat{q}_R - \widehat{q}_L \leq \pi \quad (3.26)$$

as ensured by the fact that $\widehat{\xi}$ is strictly increasing on $I_{\Lambda+\epsilon}$ and satisfies $\widehat{\xi}(x + \pi) - \widehat{\xi}(x) = (L - N)/L \leq N/L$ owing to $2N \leq L$.

Furthermore, since $\xi(\mathcal{S}_{\eta,\epsilon}(I_\Lambda))$ is an open L -independent neighbourhood of the compact interval

$$[-\nu/2; D + \nu/2] \supset \left[\frac{1}{2L}; \frac{N+1/2}{L} \right], \quad (3.27)$$

by compactness, there exists $\alpha > 0$ such that the domain $\widehat{G}(\alpha)$ as depicted in Fig. 1 satisfies $\widehat{G}(\alpha) \subset \xi(\mathcal{S}_{\eta,\epsilon}(I_\Lambda))$.

Finally, for any $a/L \in \xi(\mathcal{S}_{\eta,\epsilon}(I_\Lambda))$ it is possible to define $\widehat{x}_a = \widehat{\xi}^{-1}(a/L)$. This holds, in particular, for $a = h_p$, $p \in \llbracket 1; n \rrbracket$. Furthermore, owing to $|\lambda_a| \leq \Lambda$, if $\ell_a \in \llbracket 1; N \rrbracket$ the strict increase of $\widehat{\xi}$ on I_Λ ensures that

$$\{\lambda_a : \ell_a \in \llbracket 1; N \rrbracket\} = \{\{\widehat{x}_a\}_1^N \setminus \{\widehat{x}_{h_a}\}_1^n\}. \quad (3.28)$$

Finally, define $\{\widehat{x}_{p_a}\}_1^n = \{\lambda_a\}_1^N \setminus \{\{\widehat{x}_a\}_1^N \setminus \{\widehat{x}_{h_a}\}_1^n\}$. One needs to recourse to such a definition since it could be that $p_a/L \notin \xi(\mathcal{S}_{\eta,\epsilon}(I_\Lambda))$. However, for those $p_a/L \in \xi(\mathcal{S}_{\eta,\epsilon}(I_\Lambda))$ one does have $\widehat{x}_{p_a} = \widehat{\xi}^{-1}(p_a/L)$.

These properties being established, it follows from a straightforward computation of residues and the strict increase of $\widehat{\xi}$ on $I_{\Lambda+\epsilon}$ that

$$\widehat{\xi}(\lambda) = \frac{p(\lambda)}{2\pi} - \frac{1}{2\pi L} \sum_{a=1}^n [\theta(\lambda - \widehat{x}_{p_a}) - \theta(\lambda - \widehat{x}_{h_a})] + \frac{N+1}{2L} - \oint_{\widehat{C}} \frac{\theta(\lambda - \mu) \widehat{\xi}'(\mu)}{e^{2i\pi L \widehat{\xi}(\mu)} - 1} \cdot \frac{d\mu}{2\pi} \quad (3.29)$$

where the contour \widehat{C} is defined as $\widehat{C} = \widehat{\xi}^{-1}(\widehat{\Gamma}_+ \cup \widehat{\Gamma}_-)$ and $\widehat{\Gamma}_\pm$ have been depicted in Fig. 1. The expression can be further rearranged. Namely, setting $\widehat{C}_\epsilon = \widehat{\xi}^{-1}(\widehat{\Gamma}_\epsilon)$, one has

$$\begin{aligned} - \oint_{\widehat{C}} \frac{\theta(\lambda - \mu) \widehat{\xi}'(\mu)}{e^{2i\pi L \widehat{\xi}(\mu)} - 1} \cdot \frac{d\mu}{2\pi} &= - \int_{\widehat{q}_L}^{\widehat{q}_R} \widehat{\xi}'(\mu) \theta(\lambda - \mu) \cdot \frac{d\mu}{2\pi} + \sum_{\epsilon=\pm} \epsilon \int_{\widehat{C}_\epsilon} \frac{\theta(\lambda - \mu) \widehat{\xi}'(\mu)}{e^{-2i\pi \epsilon L \widehat{\xi}(\mu)} - 1} \cdot \frac{d\mu}{2\pi} \\ &= - \frac{N}{4\pi L} \{ \theta(\lambda - \widehat{q}_R) + \theta(\lambda - \widehat{q}_L) \} - \int_{\widehat{q}_L}^{\widehat{q}_R} K(\lambda - \mu) \widehat{\xi}_{\text{sym}}(\mu) \cdot d\mu + r_1[\widehat{\xi}](\lambda). \end{aligned} \quad (3.30)$$

In the second line, appears the function

$$\widehat{\xi}_{\text{sym}}(\lambda) = \widehat{\xi}(\lambda) - \frac{N+1}{2L} \quad (3.31)$$

and r_1 is the remainder

$$r_1[\widehat{\xi}](\lambda) = - \sum_{\epsilon=\pm} \int_{\widehat{\Gamma}_\epsilon} \frac{K(\lambda - \widehat{\xi}^{-1}(z))}{\widehat{\xi}'(\widehat{\xi}^{-1}(z))} \cdot \ln[1 - e^{2i\pi \epsilon L z}] \cdot \frac{dz}{2i\pi L}. \quad (3.32)$$

Note that the second line is obtained by carrying out an integration by parts followed by a change of variables in what concerns the form of the remainder $r_1[\widehat{\xi}](\lambda)$ given in (3.107).

Thus, all-in-all, one gets the representation :

$$\begin{aligned} \widehat{\xi}_{\text{sym}}(\lambda) + \int_{\widehat{q}_L}^{\widehat{q}_R} K(\lambda - \mu) \widehat{\xi}_{\text{sym}}(\mu) \cdot d\mu &= \frac{p(\lambda)}{2\pi} - \frac{N}{4\pi L} \{ \theta(\lambda - \widehat{q}_R) + \theta(\lambda - \widehat{q}_L) \} \\ &\quad - \frac{1}{2\pi L} \sum_{a=1}^n [\theta(\lambda - \widehat{x}_{p_a}) - \theta(\lambda - \widehat{x}_{h_a})] + r_1[\widehat{\xi}](\lambda). \end{aligned} \quad (3.33)$$

The latter is already enough so as to characterise the limit ξ . Indeed, define $q_L = \xi^{-1}(0)$ and $q_R = \xi^{-1}(D)$. Then, one has

$$\begin{aligned} |\widehat{q}_L - q_L| &\leq \left| \widehat{\xi}^{-1}\left(\frac{1}{2L}\right) - \xi^{-1}\left(\frac{1}{2L}\right) \right| + \left| \xi^{-1}\left(\frac{1}{2L}\right) - \xi^{-1}(0) \right| \\ &\leq C_1 \|\widehat{\xi} - \xi\|_{L^\infty(S_{\eta,\Delta}(I_{2\Lambda}))} + \frac{C_2}{2L} \|(\xi^{-1})'\|_{L^\infty(\xi(S_{\eta,\epsilon}(I_\Lambda)))} = o(1). \end{aligned} \quad (3.34)$$

Note that, in the second chain of bounds, I used that

$$\|\widehat{\xi}^{-1} - \xi^{-1}\|_{L^\infty(\xi(S_{\eta,\epsilon}(I_\Lambda)))} \leq C \cdot \|\widehat{\xi} - \xi\|_{L^\infty(S_{\eta,\Delta}(I_{2\Lambda}))}, \quad (3.35)$$

as ensured by Proposition A.1. Similarly to (3.34), one concludes that $|\widehat{q}_R - q_R| = o(1)$. These estimates brought together with (3.26) ensure that, for $\Delta > 1$, $q_R - q_L \leq \pi$.

It remains to bound $r_1[\widehat{\xi}](\lambda)$. For any $\lambda \in \mathbb{R}$ one has

$$|r_1[\widehat{\xi}](\lambda)| \leq \|K\|_{L^\infty(S_{2\eta}(\mathbb{R}))} \cdot \left\{ \inf_{S_{2\eta,2\epsilon}(I_\Lambda)} |\widehat{\xi}'| \right\}^{-1} \cdot \sum_{\epsilon=\pm} \int_{\widehat{\Gamma}_\epsilon} \left| \ln[1 - e^{2i\pi \epsilon L z}] \right| \cdot \frac{|dz|}{2\pi L} \leq \frac{C'}{L^2}. \quad (3.36)$$

The last bound follows from $\inf_{S_{2\eta, 2\epsilon}(I_\Lambda)} |\widehat{\xi}'| > \frac{1}{2} \inf_{S_{2\eta, 2\epsilon}(I_\Lambda)} |\xi'| > 0$ for L large enough. Also the $1/L^2$ decay in (3.36) follows after an asymptotic estimation of the integral by a variant of Watson's lemma : the boundaries of $\widehat{\Gamma}_\epsilon$ generate an algebraic decay in L starting with $O(L^{-1})$ while all other contributions to the integral are exponentially small. The uniform convergence of $\widehat{\xi}_{\text{sym}}$ to $\xi_{\text{sym}} = \xi - D/2$ and the above bounds are enough so as to take the $L \rightarrow +\infty$ on the level of (3.33). This yields the system of equation for three unknowns : the function ξ_{sym}

$$(\text{id} + \mathbf{K}_{[q_L; q_R]})(\xi_{\text{sym}})(\lambda) = \frac{p_0(\lambda)}{2\pi} - \frac{D}{4\pi} \{ \theta(\lambda - q_R) + \theta(\lambda - q_L) \} \quad (3.37)$$

and the endpoints of integration q_L, q_R

$$\xi_{\text{sym}}(q_R) = -\xi_{\text{sym}}(q_L) = \frac{D}{2}. \quad (3.38)$$

Also, the additional condition $q_R - q_L \leq \pi$ is imposed for $\Delta > 1$.

In virtue of Proposition 1.1, there exists a unique solution to (3.37)-(3.38) given by $q_R = -q_L = q$ and $\xi_{\text{sym}}(\lambda) = p(\lambda \mid q)$, with q the magnetic Fermi boundary associated with D . This ensures the bounds (3.14) and, upon elementary manipulations, the form taken by the non-linear integral equation (3.15) satisfied by $\widehat{\xi}$.

• Diffeomorphism property

Lemma 1.5 ensures that $\xi_0^{-1}([0; \widehat{D}]) = [-\widehat{q}; \widehat{q}]$. Furthermore, since $\widehat{D} \rightarrow D$ with either $D < 1/2$ for $-1 < \Delta \leq 1$ or $D \in [0; 1/2]$ for $\Delta > 1$, one has for L large-enough that $\widehat{q} < 2q < +\infty$. Then, by Proposition A.1, one readily deduces the biholomorphism property from (3.14).

Then straightforward bounds in the non-linear term $\widehat{\mathfrak{R}}'_N[\widehat{\xi}]$ based on the fact that

$$\sup_{\mu \in S_{2\eta, 2\epsilon}(I_{\widehat{q}})} \{ |\partial_\lambda R_{I_{\widehat{q}}}(\lambda, \mu)| + |R_{I_{\widehat{q}}}(\lambda, \mu)| \} \leq C \cdot g_\Delta(\lambda) \quad \text{with} \quad g_\Delta(\lambda) = \begin{cases} 1 & \text{for } \Delta > 1 \\ 1/(\lambda^2 + 1) & \text{for } \Delta = 1 \\ 1/\cosh(2\lambda) & \text{for } -1 < \Delta < 1 \end{cases} \quad (3.39)$$

ensure that

$$\widehat{\xi}'(\lambda) = \xi'_0(\lambda \mid \widehat{q}) + O\left(\frac{1}{L} g_\Delta(\lambda)\right) \quad (3.40)$$

with a remainder uniform in L and $\lambda \in \mathbb{R}$. Since g_Δ/ξ'_0 is bounded on \mathbb{R} , this ensures that, for L large enough $\widehat{\xi}' > 0$. Therefore, $\widehat{\xi}$ is a strictly increasing diffeomorphism from \mathbb{R} onto $\widehat{\xi}(\mathbb{R})$. The explicit form of the range for $-1 < \Delta \leq 1$ follows from computing the limits of $\widehat{\xi}(\lambda)$ at $\lambda \rightarrow \pm\infty$ starting from the definition (3.1) of the counting function. When $\Delta > 1$, the finite difference growth $\widehat{\xi}(x+\pi) - \widehat{\xi}(x) = (L-N)/L$ ensures that $\widehat{\xi}$ is a strictly increasing diffeomorphism from \mathbb{R} onto \mathbb{R} .

• Uniqueness of solutions to the non-linear integral equation

By hypothesis, for $-1 < \Delta \leq 1$, there exists $0 < \nu$ small enough such that $1/2 - D - \nu > 0$ and

$$\frac{p_a}{L} \in K_\nu = \left[-\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - D - \nu \right); \frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - D - \nu \right) + D \right] \quad \text{for } a = 1, \dots, n. \quad (3.41)$$

Then, by Lemma 1.5, for $-1 < \Delta \leq 1$ one has $\xi_0^{-1}(K_\nu \mid \widehat{q}) \subset I_\gamma$ for some $\gamma > 0$ and uniformly in L . Likewise, when $\Delta > 1$, given $C > 0$ such that $|p_a/L| < C$, one has $\xi_0^{-1}(I_{2C} \mid \widehat{q}) \subset I_\gamma$ for some $\gamma > 0$ and uniformly in L .

Suppose that one is give two solutions $\widehat{\xi}_1, \widehat{\xi}_2$ to the non-linear integral equation (3.15) and satisfying to all the other requirements, (3.14) in particular. In virtue of Proposition A.1, there exists $\eta, \epsilon > 0$ such that for any $z \in \xi_0(\mathcal{S}_{\eta, \epsilon}(I_\gamma) \mid \widehat{q})$, one has

$$\widehat{\xi}_a^{-1}(z) = \oint_{\partial \mathcal{S}_{2\eta, 2\epsilon}(I_\gamma)} \frac{\lambda \cdot \widehat{\xi}'_a(\lambda)}{\widehat{\xi}_a(\lambda) - z} \cdot \frac{d\lambda}{2i\pi} \quad \text{and} \quad \inf_{\lambda \in \partial \mathcal{S}_{2\eta, 2\epsilon}(I_\gamma)} \inf_{z \in \mathcal{S}_{\eta, \epsilon}(I_\gamma)} |\widehat{\xi}_a(\lambda) - z| > c_0 > 0 \quad (3.42)$$

for $a = 1, 2$ and some constant c_0 only depending on $\xi_0(* \mid q)$. Likewise, one has

$$\|\widehat{\xi}_a^{-1} - \xi_0^{-1}(* \mid \widehat{q})\|_{L^\infty(\xi_0(\mathcal{S}_{\eta, \epsilon}(I_\gamma) \mid q))} \leq C \cdot \|\widehat{\xi}_a - \xi_0(* \mid \widehat{q})\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))} \quad a = 1, 2, \quad (3.43)$$

and, after straightforward bounds in (3.42), one gets that, for some constant $C > 0$,

$$\|\widehat{\xi}_1^{-1} - \widehat{\xi}_2^{-1}\|_{L^\infty(\xi_0(\mathcal{S}_{\eta, \epsilon}(I_\gamma) \mid q))} \leq C \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))}. \quad (3.44)$$

I stress that, in virtue of the condition satisfied by the p_a 's, provided that L is large enough, for all

$$\ell \in \{h_1, \dots, h_n, p_1, \dots, p_n\} \cup [0; N+1] \quad \text{it holds} \quad \frac{\ell}{L} \in \xi_0(\mathcal{S}_{\eta, \epsilon}(I_\gamma) \mid \widehat{q}). \quad (3.45)$$

All is now in place to estimate the norm $\|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_\gamma))}$ by using the non-linear integral equation satisfied by $\widehat{\xi}_a$. Agreeing upon $\widehat{x}_{\ell, a} = \widehat{\xi}_a^{-1}(\ell/L)$, in virtue of (3.44), it holds

$$\begin{aligned} & \left\| \Phi_q^{(0)}(* \mid \{\widehat{x}_{p_a, 1}\}_1^n; \{\widehat{x}_{h_a, 1}\}_1^n) - \Phi_q^{(0)}(* \mid \{\widehat{x}_{p_a, 2}\}_1^n; \{\widehat{x}_{h_a, 2}\}_1^n) \right\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))} \\ & \leq \frac{Cn}{L} \|\partial_2 \varphi\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}) \times \mathbb{R})} \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))}. \end{aligned} \quad (3.46)$$

Further, define

$$\widehat{q}_{R, a} = \widehat{\xi}_a^{-1}\left(\frac{N+1}{2L}\right) \quad \text{and} \quad \widehat{q}_{L, a} = \widehat{\xi}_a^{-1}\left(\frac{1}{2L}\right). \quad (3.47)$$

In order to bound $\Re_{N, 2}[\widehat{\xi}](\lambda)$, it is enough to observe that, in virtue of (3.43),

$$|\widehat{q}_{R, a} - \widehat{q}| \leq \frac{C}{L} \quad \text{and} \quad |\widehat{q}_{R, 1} - \widehat{q}_{R, 2}| \leq C \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))} \leq \frac{1}{L} \cdot 2C'. \quad (3.48)$$

Then, it holds, for any $\lambda \in \mathcal{S}_{\kappa_\Delta}(I_{2\gamma})$

$$\begin{aligned} & \left| \Re_{N, 2}[\widehat{\xi}_1](\lambda) - \Re_{N, 2}[\widehat{\xi}_2](\lambda) \right| \leq \left| \int_{\widehat{q}}^{\widehat{q}_{R, 1}} R_{L, \widehat{q}}(\lambda, \mu) [\widehat{\xi}_1(\mu) - \widehat{\xi}_2(\mu)] d\mu \right| + \left| \int_{\widehat{q}_{R, 1}}^{\widehat{q}_{R, 2}} R_{L, \widehat{q}}(\lambda, \mu) [\widehat{\xi}_2(\mu) - \widehat{\xi}_2(\widehat{q}_{R, 2})] d\mu \right| \\ & \leq \|R_{L, \widehat{q}}\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}^2(I_{2\gamma}))} \cdot \left\{ |\widehat{q} - \widehat{q}_{R, 1}| \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))} + |\widehat{q}_{R, 2} - \widehat{q}_{R, 1}|^2 \|\widehat{\xi}_2'\|_{L^\infty(\mathcal{S}_{\epsilon, \eta}(I_\gamma))} \right\} \leq \frac{\widetilde{C}}{L} \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\kappa_\Delta}(I_{2\gamma}))}. \end{aligned}$$

An identical type of bound can be obtained for $\Re_{N, 3}$. Finally, using that, for L large enough and independent of a

$$\|\widehat{\xi}'_a\|_{L^\infty(\mathcal{S}_{2\eta, 2\epsilon}(I_\gamma))} \geq \frac{1}{2} \|\xi_0'(* \mid \widehat{q})\|_{L^\infty(\mathcal{S}_{2\eta, 2\epsilon}(I_\gamma))} > 0 \quad (3.49)$$

one bounds $\mathfrak{R}_{N;1}$ as

$$\|\mathfrak{R}_{N;1}[\widehat{\xi}_1] - \mathfrak{R}_{N;1}[\widehat{\xi}_2]\|_{L^\infty(\mathcal{S}_{\mathcal{N}_\Delta}(I_{2\gamma}))} \leq \frac{C'}{L} \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\mathcal{N}_\Delta}(I_{2\gamma}))}. \quad (3.50)$$

By taking the difference of the two non-linear integral equations satisfied by $\widehat{\xi}_1$ and $\widehat{\xi}_2$, the various bounds obtained earlier lead to

$$\|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\mathcal{N}_\Delta}(I_{2\gamma}))} \leq \frac{C''}{L} \cdot \|\widehat{\xi}_1 - \widehat{\xi}_2\|_{L^\infty(\mathcal{S}_{\mathcal{N}_\Delta}(I_{2\gamma}))}. \quad (3.51)$$

The latter can only hold for L large enough provided that $\widehat{\xi}_1 = \widehat{\xi}_2$, thus entailing uniqueness of solutions.

• ξ' is not necessarily positive on $[-\Lambda; \Lambda]$.

In this last part of the proof, I study the case where, *a priori* the limit ξ' of the extracted subsequence is not positive and show that such a situation cannot arise.

Prior to going into the details of the analysis, one should observe that this situation cannot arise for $-1 < \Delta \leq 0$ since then one has the trivial bound $2\pi\widehat{\xi}'(\lambda) > p'(\lambda) > 0$ on I_Λ .

When $\Delta > 1$, it is readily seen that the functions $\widehat{\xi}'$ and its limit ξ' are π periodic. In principle, one could have that $\Lambda > \pi/2$. Yet, then it is enough to observe that the Bethe roots can be decomposed as $\lambda_a = \widetilde{\lambda}_a + n_a \pi$, where $n_a \in \mathbb{Z}$ is such that $\widetilde{\lambda}_a \in]-\pi/2; \pi/2]$. Making explicit the dependence of the counting functions on the Bethe roots, it holds

$$\widehat{\xi}'(\lambda \mid \{\lambda_a\}_1^N) = \widehat{\xi}'(\lambda \mid \{\widetilde{\lambda}_a\}_1^N). \quad (3.52)$$

Therefore, since both functions will admit the same limit of the extracted sequence, it is enough to reason on the level of the $\widetilde{\lambda}_a$ which all belong to the interval $[-\pi/2; \pi/2]$. Once that it will be established that $\xi' > 0$ on $[-\pi/2; \pi/2]$, its strict positivity on intervals of large diameter will follow. Hence, below, when $\Delta > 1$, we shall assume that $0 \leq \Lambda \leq \pi/2$.

The function ξ' is holomorphic on $\mathcal{S}_{\mathcal{N}_\Delta}(I_{2\Lambda})$, and thus admits a finite number of zeroes on $[-\Lambda; \Lambda]$. Let

$$-\Lambda = \mathfrak{z}^{(0)} < \mathfrak{z}^{(1)} < \dots < \mathfrak{z}^{(r)} < \mathfrak{z}^{(r+1)} = \Lambda \quad (3.53)$$

be such that

$$\{z \in [-\Lambda; \Lambda] : \xi'(z) = 0 \text{ or } z = \pm\Lambda\} = \{\mathfrak{z}^{(a)}\}_1^{(r+1)}. \quad (3.54)$$

Also, given $\delta > 0$ and small enough, let

$$\kappa^{(k)} = \text{sgn}(\xi'_{|\mathfrak{z}^{(k)}; \mathfrak{z}^{(k+1)}[}) \quad \text{and} \quad I_\delta^{(k)} = [\mathfrak{z}^{(k)} + \delta; \mathfrak{z}^{(k+1)} - \delta] \quad \text{for } k = 0, \dots, r. \quad (3.55)$$

By Proposition A.1 there exist $\epsilon, \eta > 0$ such that

$$\xi : \mathcal{S}_{2\eta, 2\epsilon}(I_\delta^{(k)}) \rightarrow \xi(\mathcal{S}_{2\eta, 2\epsilon}(I_\delta^{(k)})) \quad (3.56)$$

is a biholomorphism that satisfies

$$\xi(\mathcal{S}_{2\eta, 2\epsilon}(I_\delta^{(k)}) \cap \mathbb{H}^\pm) = \xi(\mathcal{S}_{2\eta, 2\epsilon}(I_\delta^{(k)})) \cap \mathbb{H}^{\pm\kappa^{(k)}}, \quad \widehat{\xi}(I_\delta^{(k)}) \subset \xi(\mathcal{S}_{\eta, \epsilon}(I_\delta^{(k)})) \subset \widehat{\xi}(\mathcal{S}_{2\eta, 2\epsilon}(I_\delta^{(k)})) \quad (3.57)$$

and

$$\widehat{\xi} : \mathcal{S}_{2\eta, 2\epsilon}(I_\delta^{(k)}) \cap \widehat{\xi}^{-1}(\xi(\mathcal{S}_{\eta, \epsilon}(I_\delta^{(k)}))) \rightarrow \xi(\mathcal{S}_{\eta, \epsilon}(I_\delta^{(k)})) \quad (3.58)$$

is a biholomorphism. Furthermore, one has $\text{sgn}(\widehat{\xi}') = \kappa^{(k)}$ on $I_\delta^{(k)}$.

Let $m^{(k)}$ be the multiplicity of the zero $\mathfrak{z}^{(k)}$. By Rouché's theorem, since $\|\widehat{\xi}' - \xi'\|_{L^\infty(\mathcal{S}_{\eta, \epsilon}(I_{2\Lambda}))} \rightarrow 0$, given any $\delta > 0$, there exists L_δ such that for any $L > L_\delta$ the function $\widehat{\xi}'$ has $m^{(k)}$ zeroes, counted with multiplicities, inside of the disk $\mathcal{D}_{\mathfrak{z}^{(k)}, \delta/4}$. The distinct zeroes will be denoted by $\widehat{z}_a^{(k)}$, with $a = 1, \dots, \widehat{n}^{(k)}$. *A priori* some of these zeroes can have non-zero imaginary parts while other will be real. Thus, I assume that the zeroes are ordered in such a way that

$$\widehat{z}_1^{(k)} < \dots < \widehat{z}_{\widehat{n}_{\text{Re}}^{(k)}}^{(k)} \quad \text{and} \quad \Im(\widehat{z}_a^{(k)}) \neq 0 \quad \text{for} \quad a = \widehat{n}_{\text{Re}}^{(k)} + 1, \dots, \widehat{n}^{(k)}. \quad (3.59)$$

Further, take L large enough so that

$$\left| \widehat{\xi}(\mathfrak{z}^{(k)} + \delta) - \widehat{\xi}(\mathfrak{z}^{(k+1)} - \delta) \right| > \frac{1}{2} \left| \xi(\mathfrak{z}^{(k)} + \delta) - \xi(\mathfrak{z}^{(k+1)} - \delta) \right| > 0. \quad (3.60)$$

Then, define $\widehat{q}_{R/L}^{(k)} \in I_\delta^{(k)}$ as the two solutions to $e^{2i\pi L \widehat{\xi}(\lambda)} = -1$ such that

$$\widehat{q}_R^{(k)} - \mathfrak{z}^{(k+1)} \text{ is maximal} \quad \text{and} \quad -\mathfrak{z}^{(k)} + \widehat{q}_L^{(k)} \text{ is minimal}. \quad (3.61)$$

In virtue of (3.60), for L large enough, both $\widehat{q}_{R/L}^{(k)}$ are well-defined, exists, and are distinct. Furthermore, for any given $\delta > 0$, there exists L large enough such that

$$\sum_{k=0}^r \left\{ \left| \widehat{q}_R^{(k)} - \mathfrak{z}^{(k+1)} \right| + \left| \widehat{q}_L^{(k)} - \mathfrak{z}^{(k)} \right| \right\} \leq 4(r+1)\delta. \quad (3.62)$$

Finally, let

$$X = \left\{ x \in [-\Lambda; \Lambda] : e^{2i\pi \widehat{\xi}(x)} = 1 \right\}, \quad X^{(\text{in})} = X \cap \left\{ \bigcup_{k=0}^r I_\delta^{(r)} \right\}, \quad X^{(\text{out})} = X \setminus X^{(\text{in})}. \quad (3.63)$$

Similarly, setting $Y = \{\lambda_a\}_1^N$,

$$Y^{(\text{in})} = Y \cap X^{(\text{in})} \quad \text{and define} \quad Y^{(\text{out})} = Y \setminus Y^{(\text{in})}. \quad (3.64)$$

The purpose of the below paragraph is to estimate $\#X^{(\text{out})}$. The first step consists in estimating the number of roots in the interval $[\widehat{q}_R^{(k)}; \widehat{q}_L^{(k+1)}]$ with $k = -1, \dots, r$ under the convention that

$$\widehat{q}_R^{(-1)} = -\Lambda \quad \text{and} \quad \widehat{q}_L^{(r+1)} = \Lambda. \quad (3.65)$$

$\widehat{\xi}'$ has constant sign on each of the intervals

$$]\widehat{q}_R^{(k)}; \widehat{\mathfrak{z}}_1^{(k)}[, \quad]\widehat{\mathfrak{z}}_1^{(k)}; \widehat{\mathfrak{z}}_2^{(k)}[, \quad \dots, \quad]\widehat{\mathfrak{z}}_{\widehat{n}_{\text{Re}}^{(k)}}^{(k)}; \widehat{q}_L^{(k+1)}[. \quad (3.66)$$

Due to the ordering (3.59) of the real roots one gets the upper bound

$$\begin{aligned} \# \{ X^{(\text{out})} \cap [\widehat{q}_R^{(k)}; \widehat{q}_L^{(k+1)}] \} &\leq \left[L \left| \widehat{\xi}(\widehat{q}_R^{(k)}) - \widehat{\xi}(\widehat{\mathfrak{z}}_1^{(k)}) \right| \right] + \dots + \left[L \left| \widehat{\xi}(\widehat{\mathfrak{z}}_{\widehat{n}_{\text{Re}}^{(k)}}^{(k)}) - \widehat{\xi}(\widehat{q}_L^{(k)}) \right| \right] \\ &\leq L \cdot \|\widehat{\xi}'\|_{L^\infty([\widehat{q}_R^{(k)}; \widehat{q}_L^{(k+1)})]} \cdot |\widehat{q}_R^{(k)} - \widehat{q}_L^{(k+1)}| \leq 2L \cdot \|\widehat{\xi}'\|_{L^\infty(I_\Lambda)} \cdot |\widehat{q}_R^{(k)} - \widehat{q}_L^{(k+1)}|. \end{aligned} \quad (3.67)$$

In the first line, $[*]$ denotes the integer part. Furthermore, so as to get the last bound, L is taken large enough so that $\|\widehat{\xi}'\|_{L^\infty(I_\Lambda)} \leq 2\|\xi'\|_{L^\infty(I_\Lambda)}$. Then, summing over k , and using (3.62), one arrives to

$$\#X^{(\text{out})} \leq 8(r+1)\delta L \cdot \|\xi'\|_{L^\infty(I_\Lambda)}. \quad (3.68)$$

In order to write down a non-linear integral equation satisfied by $\widehat{\xi}'$, I still need to define auxiliary contours. Let $\widehat{G}^{(k)}(\alpha)$ and its boundary $\widehat{\Gamma}_+^{(k)} \cup \widehat{\Gamma}_-^{(k)}$ be defined as in Fig. 2. By compactness, it follows that there exists $\alpha > 0$ such that $\widehat{G}^{(k)}(\alpha) \subset \xi(\mathcal{S}_{\epsilon,\eta}(I_\delta^{(k)}))$ for any $k = 0, \dots, r$.

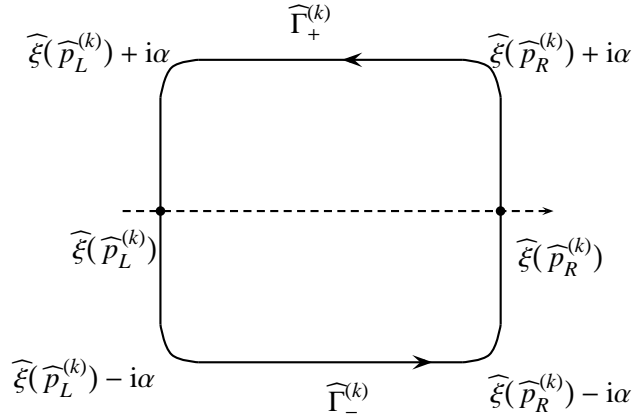


FIGURE 2 – Domain $\widehat{G}^{(k)}(\alpha)$ delimited by its boundary $\widehat{\Gamma}_+^{(k)} \cup \widehat{\Gamma}_-^{(k)}$. When $\kappa^{(k)} = 1$ one has $\widehat{p}_L^{(k)} = \widehat{q}_L^{(k)}$ and $\widehat{p}_R^{(k)} = \widehat{q}_R^{(k)}$, while, when $\kappa^{(k)} = -1$ one has $\widehat{p}_L^{(k)} = \widehat{q}_R^{(k)}$ and $\widehat{p}_R^{(k)} = \widehat{q}_L^{(k)}$.

It now remains to characterise the sum over Bethe roots occurring in $\widehat{\xi}'$. The latter can be recast as

$$\begin{aligned} -\frac{1}{L} \sum_{a=1}^N K(\lambda - \lambda_a) &= -\frac{1}{L} \sum_{y \in Y^{(\text{out})}} K(\lambda - y) - \frac{1}{L} \sum_{y \in Y^{(\text{in})}} K(\lambda - y) \\ &= -\frac{1}{L} \sum_{y \in Y^{(\text{out})}} K(\lambda - y) + \frac{1}{L} \sum_{x \in X^{(\text{in})} \setminus Y^{(\text{in})}} K(\lambda - x) - \sum_{k=0}^r \oint_{\widehat{C}^{(k)}} \frac{\widehat{\xi}'(\mu) K(\lambda - \mu)}{e^{2i\pi L \widehat{\xi}(\mu)} - 1} \cdot d\mu \end{aligned} \quad (3.69)$$

where $\widehat{C}^{(k)} = \widehat{\xi}^{-1}(\widehat{\Gamma}_+^{(k)} \cup \widehat{\Gamma}_-^{(k)})$. Now, agreeing upon $\widehat{C}_\pm^{(k)} = \widehat{C}^{(k)} \cap \mathbb{H}^\pm$, one can decompose the integral over $\widehat{C}_\pm^{(k)}$ as

$$\begin{aligned} - \oint_{\widehat{C}^{(k)}} \frac{\widehat{\xi}'(\mu) K(\lambda - \mu)}{e^{2i\pi L \widehat{\xi}(\mu)} - 1} \cdot d\mu &= -\kappa^{(k)} \int_{\widehat{q}_L^{(k)}}^{\widehat{q}_R^{(k)}} K(\lambda - \mu) \widehat{\xi}'(\mu) \cdot d\mu + \kappa^{(k)} \sum_{\epsilon = \pm} \epsilon \int_{\widehat{C}_\epsilon^{(k)}} \frac{\widehat{\xi}'(\mu) K(\lambda - \mu)}{e^{-2i\pi \epsilon \kappa_k L \widehat{\xi}(\mu)} - 1} \cdot d\mu \\ &= -\kappa^{(k)} \int_{\mathfrak{J}^{(k)}}^{\mathfrak{J}^{(k+1)}} K(\lambda - \mu) \widehat{\xi}'(\mu) \cdot d\mu + \mathfrak{r}^{(k)}[\widehat{\xi}](\lambda). \end{aligned} \quad (3.70)$$

There, I have introduced

$$r^{(k)}[\widehat{\xi}](\lambda) = -\kappa^{(k)} \left\{ \int_{\widehat{q}_L^{(k)}}^{\mathfrak{z}^{(k)}} + \int_{\mathfrak{z}^{(k+1)}}^{\widehat{q}_R^{(k)}} \right\} K(\lambda - \mu) \widehat{\xi}'(\mu) \cdot d\mu + \sum_{\epsilon=\pm} \epsilon \int_{\widehat{\Gamma}_\epsilon^{(k)}} \frac{K(\lambda - \widehat{\xi}^{-1}(z))}{e^{-2i\pi\epsilon Lz} - 1} \cdot dz. \quad (3.71)$$

Finally, introduce the intervals

$$J^{(\pm)} = \bigcup_{k: \kappa^{(k)}=\pm 1}]\mathfrak{z}^{(k)}; \mathfrak{z}^{(k+1)}[. \quad (3.72)$$

All the above leads to the non-linear integral equation satisfied by $\widehat{\xi}$

$$(\text{id} + K_{J^{(+)}} - K_{J^{(-)}})[\widehat{\xi}](\lambda) = \frac{p'(\lambda)}{2\pi} - \widehat{\phi}_{\text{out}}(\lambda) + \widehat{\phi}_{\text{in}}(\lambda) + \sum_{k=0}^r r^{(k)}[\widehat{\xi}](\lambda). \quad (3.73)$$

There, I have set

$$\widehat{\phi}_{\text{out}}(\lambda) = \frac{1}{L} \sum_{y \in Y^{(\text{out})}} K(\lambda - y) \quad \text{and} \quad \widehat{\phi}_{\text{in}}(\lambda) = \frac{1}{L} \sum_{x \in X^{(\text{in})} \setminus Y^{(\text{in})}} K(\lambda - x). \quad (3.74)$$

Let $R_{J^{(+)}}$ be the resolvent operator ‡ to $\text{id} + K_{J^{(+)}}$. By Lemma 1.3, the resolvent kernel is strictly positive : $R_{J^{(+)}}(\lambda, \mu) > 0$. Furthermore, by (1.16)

$$P'_{J^{(+)}}(\lambda) = (\text{id} - R_{J^{(+)}}) \left[\frac{p'}{2\pi} \right](\lambda) \geq (\text{id} - L_{\mathbb{R}}) \left[\frac{p'}{2\pi} \right](\lambda) = \rho_{\infty}(\lambda) > 0 \quad \text{on } \mathbb{R}. \quad (3.75)$$

Thus, adopting the notations of Lemma 1.4, one gets

$$(\text{id} - [R_{J^{(+)}}]_{J^{(-)}})[\widehat{\xi}](\lambda) = P'_{J^{(+)}}(\lambda) - \widehat{\psi}_{\text{out}}(\lambda) + \widehat{\psi}_{\text{in}}(\lambda) + \sum_{k=0}^r \widetilde{r}^{(k)}[\widehat{\xi}](\lambda) \quad (3.76)$$

where

$$\widehat{\psi}_{\text{out}}(\lambda) = \frac{1}{L} \sum_{y \in Y^{(\text{out})}} R_{J^{(+)}}(\lambda, y) \quad \text{and} \quad \widehat{\psi}_{\text{in}}(\lambda) = \frac{1}{L} \sum_{x \in X^{(\text{in})} \setminus Y^{(\text{in})}} R_{J^{(+)}}(\lambda, x). \quad (3.77)$$

Finally,

$$\widetilde{r}^{(k)}[\widehat{\xi}](\lambda) = -\kappa^{(k)} \left\{ \int_{\widehat{q}_L^{(k)}}^{\mathfrak{z}^{(k)}} + \int_{\mathfrak{z}^{(k+1)}}^{\widehat{q}_R^{(k)}} \right\} R_{J^{(+)}}(\lambda, \mu) \widehat{\xi}'(\mu) \cdot d\mu + \sum_{\epsilon=\pm} \epsilon \int_{\widehat{\Gamma}_\epsilon^{(k)}} \frac{R_{J^{(+)}}(\lambda, \widehat{\xi}^{-1}(z))}{e^{2i\pi\epsilon Lz} - 1} \cdot dz. \quad (3.78)$$

It follows from Lemma 1.4 that the operator $\text{id} - [R_{J^{(+)}}]_{J^{(-)}}$ is invertible and that its resolvent operator $[\mathcal{R}_{J^{(+)}}]_{J^{(-)}}$ has a strictly positive integral kernel $[\mathcal{R}_{J^{(+)}}]_{J^{(-)}}(\lambda, \mu) > 0$. Therefore, one has

$$\widehat{\xi}'(\lambda) = (\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}}) \left[P'_{J^{(+)}} - \widehat{\psi}_{\text{out}} + \widehat{\psi}_{\text{in}} + \sum_{k=0}^r \widetilde{r}^{(k)}[\widehat{\xi}] \right](\lambda). \quad (3.79)$$

\ddagger . Recall that when $\Delta > 1$ it holds $\text{diam}(J^{(+)} \cup J^{(-)}) < \pi$, so that indeed the involved operators are invertible in virtue of Lemma 1.3

By the above, the operators $\mathcal{R}_{J^{(-)}}$ and *a fortiori* $\text{id} + \mathcal{R}_{J^{(-)}}$ are strictly positive. Since, $P'_{J^{(+)}}(\lambda) > 0$, $\widehat{\psi}_{\text{in}}(\lambda) \geq 0$, this leads to the lower bound

$$\widehat{\xi}'(\lambda) \geq P'_{J^{(+)}}(\lambda) + \left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right) \left[-\widehat{\psi}_{\text{out}} + \sum_{k=0}^r \widetilde{\mathbf{r}}^{(k)}[\widehat{\xi}] \right](\lambda). \quad (3.80)$$

Since $\#\{Y^{(\text{out})} \setminus X^{(\text{out})}\} \leq n$ and $\#\{Y^{(\text{out})} \cap X^{(\text{out})}\}$ can be bounded by $\#X^{(\text{out})}$, the bound (3.68) ensures that

$$\left| \widehat{\psi}_{\text{out}}(\lambda) \right| \leq \|R_{J^{(+)}}\|_{L^\infty(\mathbb{R}^2)} \frac{\#Y^{(\text{out})}}{2\pi L} \leq C\left(\delta + \frac{n}{L}\right) \quad \text{leading to} \quad \limsup_{L \rightarrow +\infty} \left| \left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right) \left[-\widehat{\psi}_{\text{out}} \right](\lambda) \right| \leq C'\delta \quad (3.81)$$

by continuity of $\left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right)$ and with a bound that is uniform on \mathbb{R} .

Owing to $\|\widehat{\xi}\|_{L^\infty(I_\Lambda)} \leq 2\|\xi\|_{L^\infty(I_\Lambda)}$ and $\widehat{q}_{R/L}^{(k)} \in [-\Lambda; \Lambda]$, it follows readily that the contours $\widehat{\Gamma}_\epsilon^{(k)}$ are all bounded in L and thus for any $\lambda \in I_\Lambda$

$$\left| \sum_{k=0}^r \sum_{\epsilon=\pm} \epsilon \int_{\widehat{\Gamma}_\epsilon^{(k)}} \frac{R_{J^{(+)}}(\lambda, \widehat{\xi}^{-1}(z))}{e^{2i\pi\epsilon Lz} - 1} \cdot dz \right| \leq \|R_{J^{(+)}}\|_{L^\infty(I_\Lambda \times \mathcal{S}_{\kappa_\Delta}(\mathbb{R}))} \cdot \frac{C}{L} \quad (3.82)$$

where the $1/L$ decay follows from a straightforward estimation of the integral while the bound on the integral kernel $R_{J^{(+)}}$ follows from the inclusion $\widehat{\Gamma}_\pm^{(k)} \subset \xi(\mathcal{S}_{\eta,\epsilon}(I_\delta^{(k)}))$ what leads to that $\widehat{\xi}^{-1}(\widehat{\Gamma}_\pm^{(k)}) \subset \mathcal{S}_{2\eta,2\epsilon}(I_\delta^{(k)}) \subset \mathcal{S}_{\kappa_\Delta}(I_{\Lambda+2\epsilon})$ as ensured by (3.57). Thus, in virtue of (3.62), one gets

$$\limsup_{L \rightarrow +\infty} \left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right) \left[\sum_{k=0}^r \widetilde{\mathbf{r}}^{(k)}[\widehat{\xi}] \right](\lambda) = O(\delta). \quad (3.83)$$

This being settled, it remains to take the $L \rightarrow +\infty$ limit superior of the inequality (3.80) followed by sending $\delta \rightarrow 0^+$. One gets, for any $\lambda \in I_\Lambda$

$$\xi'(\lambda) \geq P'_{J^{(+)}}(\lambda) \geq \rho_\infty(\lambda) > \inf_{I_\Lambda} \rho_\infty > 0. \quad (3.84)$$

This contradicts that, either $\xi'(\lambda) < 0$ on I_Λ or that ξ' has zeroes on I_Λ . ■

3.2 Various corollaries of interest

Theorem 3.2 has several important corollaries. First of all, it guarantees the uniqueness of solutions to the logarithmic Bethe Ansatz equations (0.7) throughout the regime $\Delta > -1$ and, in particular, for $\Delta > 0$ when convexity arguments cannot be used. To the best of my knowledge, uniqueness of such solutions has never been proven earlier.

Corollary 3.3. *Under the hypothesis of Theorem 3.2 and point v) in particular, there exists L_0 such that the system of logarithmic Bethe Ansatz equations associated with the given choice of integers $\{\ell_a\}_1^N$ admits a unique real-valued solution for any $L \geq L_0$.*

Proof —

If there would be two solutions to the logarithmic Bethe equations associated with the given choice of integers $\{\ell_a\}_1^N$, than one would then be able to build two distinct counting functions $\widehat{\xi}^{(a)}$, $a = 1, 2$. These, however, will both satisfy the non-linear integral equation when L will be large enough. As follows from Theorem 3.2 point v), the non-linear integral equation admits a unique solution, contradicting that $\widehat{\xi}^{(1)} \neq \widehat{\xi}^{(2)}$ for L large enough. ■

A second important consequence of Theorem 3.2 is the existence of an all order large- L asymptotic expansion of the counting function.

Proposition 3.4. Let $h_1 < \dots < h_n$, $h_a \in \llbracket 1; N' \rrbracket$ and $p_1 < \dots < p_n$, $p_a \in \mathbb{Z} \setminus \llbracket 1; N' \rrbracket$ be and increasing sequence of integers such that

- for $-1 < \Delta \leq 1$, $\Delta = \cos(\zeta)$,

$$\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N' - 1}{L} \right) > \frac{p_n - N'}{L} \quad , \quad \frac{p_1 - 1}{L} > -\frac{\pi - \zeta}{\pi} \left(\frac{1}{2} - \frac{N' - 1}{L} \right) ; \quad (3.85)$$

- for $\Delta > 1$, the integers $\{p_a\}_1^n$ are uniformly bounded in L : $|p_a/L| \leq C$ for some L -independent $C > 0$.

Let $\{\lambda_a\}_1^{N'}$ be the solution to the logarithmic Bethe equations subordinate to this choice of integers and let $N' = N + s$ with s bounded uniformly in L . Then, the associated counting function $\widehat{\xi}$ admits the large- L asymptotic expansion

$$\widehat{\xi}(\lambda) = \xi_0(\lambda | \widehat{q}) + \sum_{p=1}^r \frac{1}{L^p} \xi_p^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) + O(L^{-(r+1)}) . \quad (3.86)$$

There \widehat{x}_a are defined through $\widehat{\xi}(\widehat{x}_a) = a/L$.

This asymptotic expansion is such that the remainder just as all functions, ξ_p , $p = 0, \dots, r$ are holomorphic on $\mathcal{S}_{\zeta/2}(\mathbb{R})$. Furthermore, the remainder is uniform on $\overline{\mathcal{S}}_\eta$ for any $0 < \eta < \zeta/2$. The function ξ_0 is as defined in (1.51) with \widehat{q} the solution to the magnetic Fermi boundary problem associated with $\widehat{D} = N/L$. Also,

$$\xi_1^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) = \Phi_{\widehat{q}}^{(s)}(\lambda | \{\widehat{x}_{p'_a}\}_1^{n+s}; \{\widehat{x}_{h_a}\}_1^n) \quad \text{where} \quad \{p'_a\}_1^{n+s} = \{p_a\}_1^{n+s} \cup \{a\}_1^s, \quad (3.87)$$

$\Phi_{\widehat{q}}^{(s)}$ given in (3.17) and

$$\xi_2^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) = \frac{1}{2\xi'_0(\widehat{q} | \widehat{q})} \sum_{\epsilon=\pm} \epsilon R_{I_{\widehat{q}}}(\lambda, \epsilon \widehat{q}) \cdot \left[\left(\xi_1^{(s)}(\epsilon \widehat{q} | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) - \frac{1}{2} \right)^2 - \frac{1}{12} \right]. \quad (3.88)$$

Finally, the unique solutions $\widehat{q}_{L/R}$ to $\widehat{\xi}(\widehat{q}_L) = 1/(2L)$ and $\widehat{\xi}(\widehat{q}_R) = (N + 1/2)/L$ admit the large- L asymptotic expansion

$$\widehat{q}_R - \widehat{q} = \sum_{k=1}^r \frac{q_+^{(k)}}{L^k} + O\left(\frac{1}{L^{r+1}}\right) \quad \text{and} \quad \widehat{q}_L + \widehat{q} = \sum_{k=1}^r \frac{q_-^{(k)}}{L^k} + O\left(\frac{1}{L^{r+1}}\right) \quad (3.89)$$

where the first terms of the expansion take the form

$$q_{\pm}^{(1)} = \frac{1/2 - \xi_1^{(s)}(\pm \widehat{q})}{\xi'_0(\widehat{q} | \widehat{q})} . \quad (3.90)$$

Also, the second order deviations in respect to \widehat{q} read

$$q_{\pm}^{(2)} = -\frac{1}{(\xi'_0(\widehat{q} | \widehat{q}))^2} \left\{ [1/2 - \xi_1^{(s)}(\pm \widehat{q})] (\xi_1^{(s)})'(\pm \widehat{q}) + \xi_2^{(s)}(\pm \widehat{q}) \xi'_0(\widehat{q} | \widehat{q}) + \frac{\xi''_0(\pm \widehat{q} | \widehat{q})}{2\xi'_0(\pm \widehat{q} | \widehat{q})} [1/2 - \xi_1^{(s)}(\pm \widehat{q})]^2 \right\} . \quad (3.91)$$

The dependence of $\xi_1^{(s)}$ and $\xi_2^{(s)}$ on the auxiliary rapidities has been omitted both in (3.90) and (3.91).

The decomposition $N' = N + s$ might appear slightly artificial in that one can absorb all the dependence on s by simply doing the substitution $(\widehat{D}, \widehat{q}, s) \hookrightarrow (\widehat{D}', \widehat{q}', 0)$ where $\widehat{D}' = N'/L$ and \widehat{q}' is the magnetic Fermi boundary

associated with \widehat{D}' . However presenting the expansion in the above form allows one to immediately take into account various terms of the asymptotic expansion when $\widehat{D} \rightarrow D$ sufficiently fast in L for instance

$$\widehat{D} - D = O\left(\frac{1}{L^{r+1}}\right) \quad \text{while} \quad \widehat{D}' - D = O\left(\frac{1}{L}\right). \quad (3.92)$$

Further, note that all the terms of the asymptotic expansion depend implicitly on \widehat{q} . It is readily seen that the latter admits an all order asymptotic expansion in $D - \widehat{D}$. Presenting the asymptotic expansion of $\widehat{\xi}$ in the form (3.97) allows one to absorb all the orders in $D - \widehat{D}$ and thus have a slightly simpler to obtain, from the computational point of view, asymptotic expansion. If one would make the additional hypothesis $\widehat{D} - D = O(L^{-(r+1)})$ then one would also have $\widehat{q} - q = O(L^{-(r+1)})$ and it would be possible to simplify (3.97) further by replacing \widehat{q} by q . The presence of terms $\widehat{D} - D$ in the asymptotic expansion was first noticed in [34].

Proof — The first two terms of the asymptotic expansion, just as its form, are readily obtained by recasting the non-linear integral equation (3.15) in the form

$$\widehat{\xi}(\lambda) = \widehat{\xi}_0(\lambda | \widehat{q}) + \Phi_{\widehat{q}}^{(s)}(\lambda | \{\widehat{x}_{p_a'}\}_1^{n+s}; \{\widehat{x}_{h_a}\}_1^n) + \mathfrak{R}_N[\widehat{\xi}](\lambda) \quad (3.93)$$

and then using straightforward bounds on the "remainder" operator \mathfrak{R}_N . The starting point for pushing the asymptotic expansion one order further, viz. up to $r = 2$, is to observe that the properties of the solution to the non-linear integral equation lead to an overall estimate

$$|\widehat{q} - \widehat{q}_R| + |\widehat{q} - \widehat{q}_L| = O(1/L). \quad (3.94)$$

A straightforward application of Watson's lemma to $\mathfrak{R}_{N;1}[\widehat{\xi}]$ yields

$$\begin{aligned} \mathfrak{R}_{N;1}[\widehat{\xi}](\lambda) &= -\frac{1}{4\pi^2 L^2} \left\{ \frac{R_{I_{\widehat{q}}}(\lambda, \widehat{q}_R)}{\widehat{\xi}'(\widehat{q}_R)} - \frac{R_{I_{\widehat{q}}}(\lambda, \widehat{q}_L)}{\widehat{\xi}'(\widehat{q}_L)} \right\} \int_{\mathbb{R}} \ln(1 + e^{-|t|}) \cdot dt + O\left(\frac{1}{L^3}\right) \\ &= -\frac{1}{24L^2 \widehat{\xi}'_0(\widehat{q} | \widehat{q})} (R_{I_{\widehat{q}}}(\lambda, \widehat{q}) - R_{I_{\widehat{q}}}(\lambda, -\widehat{q})) + O\left(\frac{1}{L^3}\right). \end{aligned} \quad (3.95)$$

Also, due to (3.94), it holds that

$$\mathfrak{R}_{N;2}[\widehat{\xi}](\lambda) + \mathfrak{R}_{N;3}[\widehat{\xi}](\lambda) = R_{I_{\widehat{q}}}(\lambda, \widehat{q}_R) \widehat{\xi}'(\widehat{q}_R) \frac{(\widehat{q}_R - \widehat{q})^2}{2} - R_{I_{\widehat{q}}}(\lambda, \widehat{q}_L) \widehat{\xi}'(\widehat{q}_L) \frac{(\widehat{q} + \widehat{q}_L)^2}{2} + O\left(\frac{1}{L^3}\right) \quad (3.96)$$

The above bounds already ensure that $\widehat{\xi}$ admits the large- L asymptotic expansion up to $O(L^{-3})$:

$$\widehat{\xi}(\lambda) = \xi_0(\lambda | \widehat{q}) + \frac{1}{L} \xi_1^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) + \frac{1}{L^2} \widetilde{\xi}_2^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) + O(L^{-3}) \quad (3.97)$$

where ξ_0 and $\xi_1^{(s)}$ are as defined in the statement of the proposition whereas the expression for $\widetilde{\xi}_2^{(s)}$ involves the L -dependent endpoints $\widehat{q}_{R/L}$:

$$\widetilde{\xi}_2^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) = -\sum_{\epsilon=\pm} \epsilon \frac{R_{I_{\widehat{q}}}(\lambda, \epsilon \widehat{q})}{24 \widehat{\xi}'_0(\widehat{q} | \widehat{q})} + \frac{\xi'_0(\widehat{q} | \widehat{q})}{2} \{R_{I_{\widehat{q}}}(\lambda, \widehat{q})(\widehat{q}_R - \widehat{q})^2 - R_{I_{\widehat{q}}}(\lambda, -\widehat{q})(\widehat{q} + \widehat{q}_L)^2\}. \quad (3.98)$$

To conclude, it solely remains to obtain the asymptotic expansion of $\widehat{q}_R - \widehat{q}$ and $\widehat{q}_L + \widehat{q}$ up to $O(L^{-3})$. The latter can be obtained by expanding the defining relation

$$\frac{1}{2L} = \widehat{\xi}(\widehat{q}_L) \quad \text{and} \quad \frac{N+1/2}{L} = \widehat{\xi}(\widehat{q}_R) \quad (3.99)$$

to the second order in $\widehat{q}_R - \widehat{q}$ and $\widehat{q}_L + \widehat{q}$ and using the *a priori* estimates (3.94) so as to separate slower and faster decaying terms. All-in-all one obtains the expansion (3.89) up to the second order, *viz.* $r = 2$, where $q_{\pm}^{(1)}$ is as defined in (3.90). However, at this stage of the analysis, the expression for $q_{\pm}^{(2)}$ has still not its final form in that it is given in terms of $\widetilde{\xi}_2^{(s)}$. To conclude, one should first insert the expansion (3.89) to the first order in $1/L$ into (3.98) so as to get (3.97) with $r = 2$ along with the form of the coefficients (3.87)-(3.88). Starting from there, one readily obtains the claimed form of the second order coefficient $q_{\pm}^{(2)}$ as given by (3.91).

The existence of the all order asymptotic expansion is obtained by a classical bootstrap argument, which I shall not detail here. I refer to Section 3.2 of [28] for similar handlings. ■

3.3 The counting function when $D = 1/2$ and $-1 < \Delta \leq 1$

Theorem 3.5. *Let $-1 < \Delta \leq 1$ and $N/L = D = 1/2$. Then, for any $\epsilon > 0$ there exists L_0 such that the counting function $\widehat{\xi}$ constructed from the Bethe roots associated with the choice of integers $\ell_a = a$ in the logarithmic Bethe equations admits the expansion*

$$\widehat{\xi}(\lambda) = \xi_0(\lambda | +\infty) + O(\epsilon) \quad \text{with} \quad \xi_0(\lambda | +\infty) = \int_0^\lambda \rho_\infty(\mu) \cdot d\mu \quad (3.100)$$

where the remainder is uniform in $L \geq L_0$.

Proof —

Let $\{\lambda_a\}$ be a solution to the logarithmic Bethe equations associated with the choice of integers $\ell_a = a$. In virtue of Proposition 2.3, there exists Λ_ϵ such that $\widehat{c}_{\Lambda_\epsilon} < \epsilon$, where \widehat{c}_Λ has been defined in (2.56).

The counting function $\widehat{\xi}$ associated with these roots is a sequence of holomorphic function on $S_{\times\Delta}(\mathbb{R})$ and, as such, admits a converging subsequence to some ξ . It remains to characterise the limit ξ and hence the limit of $\widehat{\xi}$ according to the reasoning of the proof of Theorem 3.2.

The first stage of the proof follows the analysis developed in the proof of Theorem 3.2 relative to waiving-off the possibility that ξ has several zeroes on \mathbb{R} . Taking the same definition of (3.63) with Λ now being replaced by Λ_ϵ one arrives to (3.79) with $\widehat{\psi}_{\text{in/out}}$ defined exactly as in (3.77). Again, by positivity of the integral operator and by invoking (3.75), one gets the lower bound (3.80), *viz.*

$$\widehat{\xi}'(\lambda) \geq \rho_\infty(\lambda) + \left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right) \left[-\widehat{\psi}_{\text{out}} + \sum_{k=0}^r \widetilde{\tau}^{(k)}[\widehat{\xi}] \right](\lambda). \quad (3.101)$$

By continuity of $\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}}$, it follows from $\widehat{c}_{\Lambda_\epsilon} < \epsilon$ and the bounds (3.68) on $\#X^{(\text{out})}$ which allow one to control the cardinality of $Y^{(\text{out})} \cap X^{(\text{out})}$ that

$$\limsup_{L \rightarrow +\infty} \left| \left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right) \left[-\widehat{\psi}_{\text{out}} \right](\lambda) \right| \leq C \cdot \left\{ \delta(r+1) \|\xi'\|_{L^\infty(I_{\Lambda_\epsilon})} + \epsilon \right\}. \quad (3.102)$$

In the above formula, r refers to the number of zeroes of ξ' on $] -\Lambda_\epsilon; \Lambda_\epsilon[$ and thus depends *a priori* on ϵ . One also obtains a similar bound on the second term, namely

$$\limsup_{L \rightarrow +\infty} \left| \left(\text{id} + [\mathcal{R}_{J^{(+)}}]_{J^{(-)}} \right) \left[\sum_{k=0}^r \widetilde{\tau}^{(k)}[\widehat{\xi}] \right](\lambda) \right| \leq C' \cdot \delta \cdot (r+1) \cdot \|\xi'\|_{L^\infty(I_{\Lambda_\epsilon})}. \quad (3.103)$$

Taking the pointwise in λ $L \rightarrow +\infty$ limit superior of (3.101), one finds the lower bound

$$\xi'(\lambda) \geq \rho_\infty(\lambda) - C'' \left(\delta \cdot (r+1) \cdot \|\xi'\|_{L^\infty(I_{\Lambda_\epsilon})} + \epsilon \right). \quad (3.104)$$

It then remains to send first $\delta \rightarrow 0^+$ and then $\epsilon \rightarrow 0^+$ so as to get the strict positivity of ξ on \mathbb{R} .

Now, taking for granted that $\xi' > 0$ on \mathbb{R} , one again picks $\epsilon > 0$ and Λ_ϵ such that $\widehat{c}_{\Lambda_\epsilon} < \epsilon$ and defines $\widehat{q}_{R;\epsilon}$, resp. $\widehat{q}_{L;\epsilon}$, to be the closed to Λ_ϵ , resp. $-\Lambda_\epsilon$, solution to $\exp\{2i\pi L\widehat{\xi}(\lambda)\} = -1$ lying outside of I_{Λ_ϵ} . Repeating similar handlings to the ones carried out in the corresponding section of the proof of Theorem 3.2, one gets

$$\begin{aligned} \widehat{\xi}_{\text{sym}}(\lambda) + \int_{\widehat{q}_{L;\epsilon}}^{\widehat{q}_{R;\epsilon}} K(\lambda - \mu) \widehat{\xi}_{\text{sym}}(\mu) \cdot d\mu &= \frac{p(\lambda)}{2\pi} \\ &- \frac{1}{2\pi} \left\{ \widehat{\xi}_{\text{sym}}(\widehat{q}_{R;\epsilon}) \cdot \theta(\lambda - \widehat{q}_{R;\epsilon}) - \widehat{\xi}_{\text{sym}}(\widehat{q}_{L;\epsilon}) \cdot \theta(\lambda - \widehat{q}_{L;\epsilon}) \right\} + r_1[\widehat{\xi}](\lambda) - \frac{1}{2\pi L} \sum_{x \in \widetilde{Y}^{(\text{out})}} \theta(\lambda - x). \end{aligned} \quad (3.105)$$

Here $\widetilde{Y}^{(\text{out})} = \{\lambda_a : \lambda_a \notin [\widehat{q}_{L;\delta}; \widehat{q}_{R;\delta}]\}$ and $r_1[\widehat{\xi}](\lambda)$ is as given by (3.107) relatively to the above defined endpoints of integration $\widehat{q}_{R/L;\epsilon}$. After a few handlings, one recasts the above equation as

$$\begin{aligned} \widehat{\xi}_{\text{sym}}(\lambda) &= p(\lambda \mid +\infty) + Z(\lambda \mid +\infty) \frac{\pi - 2\zeta}{2\pi} \cdot \left\{ \widehat{\xi}_{\text{sym}}(\widehat{q}_{R;\epsilon}) + \widehat{\xi}_{\text{sym}}(\widehat{q}_{L;\epsilon}) \right\} \\ &+ \mathfrak{R}_{N;1}^{(\text{tot})}[\widehat{\xi}](\lambda) - \frac{1}{L} \sum_{x \in \widetilde{Y}^{(\text{out})}} \varphi(\lambda - x) \end{aligned} \quad (3.106)$$

where

$$\begin{aligned} \mathfrak{R}_{N;1}^{(\text{tot})}[\widehat{\xi}](\lambda) &= \mathfrak{R}_{N;1}[\widehat{\xi}](\lambda) + \int_{\widehat{q}_{R;\epsilon}}^{+\infty} R(\lambda - \mu) [\widehat{\xi}_{\text{sym}}(\mu) - \widehat{\xi}_{\text{sym}}(\widehat{q}_{R;\epsilon})] \cdot d\mu \\ &+ \int_{-\infty}^{\widehat{q}_{L;\epsilon}} R(\lambda - \mu) [\widehat{\xi}_{\text{sym}}(\mu) - \widehat{\xi}_{\text{sym}}(\widehat{q}_{L;\epsilon})] \cdot d\mu \end{aligned} \quad (3.107)$$

and

$$\mathfrak{R}_{N;1}[\widehat{\xi}](\lambda) = - \sum_{\epsilon=\pm} \int_{\widehat{\Gamma}_\epsilon} \frac{R(\lambda - \widehat{\xi}^{-1}(z))}{\widehat{\xi}'(\widehat{\xi}^{-1}(z))} \cdot \ln[1 - e^{2i\pi\epsilon Lz}] \cdot \frac{dz}{2i\pi L}. \quad (3.108)$$

It remains to bound the various terms in (3.106). A counting of solution argument ensures that

$$\left[L(\widehat{\xi}_{\text{sym}}(-\Lambda_\epsilon) - \widehat{\xi}_{\text{sym}}(-\infty)) \right] \leq L\widehat{c}_{\Lambda_\epsilon} \quad \text{and} \quad \left[L(\widehat{\xi}_{\text{sym}}(+\infty) - \widehat{\xi}_{\text{sym}}(\Lambda_\epsilon)) \right] \leq L\widehat{c}_{\Lambda_\epsilon}. \quad (3.109)$$

Thus,

$$\widehat{\xi}_{\text{sym}}(\widehat{q}_{L;\epsilon}) - \widehat{\xi}_{\text{sym}}(-\infty) \leq \widehat{\xi}_{\text{sym}}(-\Lambda_\epsilon) - \widehat{\xi}_{\text{sym}}(-\infty) \leq \widehat{c}_{\Lambda_\epsilon} + 1/L \quad (3.110)$$

and a similar bound holds for $\widehat{\xi}_{\text{sym}}(+\infty) - \widehat{\xi}_{\text{sym}}(\widehat{q}_{R;\epsilon})$. From there, since $\xi_{\text{sym}}(+\infty) + \xi_{\text{sym}}(-\infty) = 0$, one readily bounds the second terms in (3.106). From the above bounds one also infers that

$$\left| \int_{\widehat{q}_{R;\epsilon}}^{+\infty} R(\lambda - \mu) [\widehat{\xi}_{\text{sym}}(\mu) - \widehat{\xi}_{\text{sym}}(\widehat{q}_{R;\epsilon})] \cdot d\mu \right| \leq \|R\|_{L^1(\mathbb{R})} \cdot (\widehat{c}_{\Lambda_\epsilon} + 1/L) \quad (3.111)$$

and analogously for its counterpart related to $\widehat{q}_{R;\epsilon}$. Finally, one gets that

$$\left| \Re_{N;1}[\widehat{\xi}](\lambda) \right| \leq \frac{C}{L^2} \cdot \frac{(2\Lambda_\epsilon + 1) \cdot \|R\|_{L^\infty(\mathcal{S}_\eta(\mathbb{R}))}}{\inf \left\{ \widehat{\xi}'(s) : s \in \mathcal{S}_\eta(I_{\Lambda_\epsilon}) \right\}}. \quad (3.112)$$

Given $\epsilon > 0$ fixed, the infimum is bounded from below due to the convergence of $\widehat{\xi}$ on compact subsets. This entails the claim, upon taking L large-enough. \blacksquare

4 Applications

In this section I establish several more or less direct applications of the existence of the large- L asymptotic expansion of the counting function. The first of these corresponds to the proof of the existence of limits of the type (0.13). The second application concerns the proof of the conformal behaviour of the spectrum of low-lying excitations above the ground state in the so-called massless regime of the model, namely when there exist zero energy excitations. This regime corresponds basically to $D \in]0; 1/2[$ for any $\Delta > -1$ and $D = 1/2$ for $-1 < \Delta \leq 1$. Below, I shall confine myself to the regime $0 < D < 1/2$ since the latter is technically much simpler to deal with.

4.1 Densification of Bethe roots

Theorem 4.1. *Let $N, L \rightarrow +\infty$ in such a way that $N/L \rightarrow D \in]0; 1/2]$. Let q be the unique solution to (1.25) subordinate to D and let the particle-hole integers (2.3) satisfy to the hypothesis of Theorem 3.2. Then, given any bounded Lipschitz function f on \mathbb{R} with Lipschitz constant $\text{Lip}[f]$, it holds*

$$\frac{1}{L} \sum_{a=1}^L f(\lambda_a) \rightarrow \int_{-q}^q f(\lambda) \rho(\lambda | q) \cdot d\lambda \quad (4.1)$$

where $\{\lambda_a\}_1^N$ correspond to the unique solution to the logarithmic Bethe equations subordinates to the given choice of particle-hole integers.

Proof —

Due to the unbounded nature of the Bethe roots at $D = 1/2$ and $-1 < \Delta \leq 1$, I first deal with the simpler case of a uniformly bounded in L distribution, namely when $0 \leq D < 1/2$ and $-1 < \Delta \leq 1$ or $0 \leq D \leq 1/2$ when $\Delta > 1$.

For this range of D and Δ , Proposition 2.2 ensures that the Bethe roots such that $\widehat{\xi}(\lambda_a) = a/L$ all belong to the compact $[-\Lambda; \Lambda]$. Also, it has already been established that, provided L is large enough

$$\|\widehat{\xi}^{-1} - \xi_0^{-1}(* | \widehat{q})\|_{L^\infty(\xi_0(\mathcal{S}_{\eta;\epsilon}(I_\Lambda))} \leq \|\widehat{\xi} - \xi_0(* | \widehat{q})\|_{L^\infty(\mathcal{S}_{\eta;\epsilon}(I_{2\Lambda}))} \leq \frac{C}{L} \quad (4.2)$$

for some $\epsilon, \eta > 0$ small enough. Finally, for L large enough, it holds $a/L \in \xi_0(I_{\Lambda+\epsilon} | \widehat{q})$ for any $a \in \llbracket 1; N \rrbracket$. The finite sum one starts with can be decomposed as

$$\frac{1}{L} \sum_{a=1}^N f(\lambda_a) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 \quad \text{with} \quad \mathcal{S}_1 = \frac{1}{L} \sum_{a=1}^N \left[f \circ \widehat{\xi}^{-1}\left(\frac{a}{L}\right) - f \circ \xi_0^{-1}\left(\frac{a}{L} | \widehat{q}\right) \right] \quad (4.3)$$

while

$$\mathcal{S}_2 = \frac{1}{L} \sum_{a=1}^n \left[f(\widehat{x}_{p_a}) - f(\widehat{x}_{h_a}) \right] \quad , \quad \mathcal{S}_3 = \frac{1}{L} \sum_{a=1}^N \left[f \circ \xi_0^{-1}\left(\frac{a}{L} | \widehat{q}\right) - f \circ \xi_0^{-1}\left(\frac{a}{L} | q\right) \right] \quad (4.4)$$

and

$$\mathcal{S}_4 = \frac{1}{L} \sum_{a=1}^N f \circ \xi_0^{-1} \left(\frac{a}{L} \mid q \right). \quad (4.5)$$

\mathcal{S}_4 is a Riemann sum and, as such, converges

$$\mathcal{S}_4 \rightarrow \int_0^D f \circ \xi_0^{-1}(s \mid q) \cdot ds = \int_{-q}^q f(\lambda) \rho(\lambda \mid q) \cdot d\lambda \quad (4.6)$$

where the last equality follows from $\xi_0([-q; q] \mid q) = [0; D]$ and $\xi'_0(\lambda \mid q) = \rho(\lambda \mid q)$. Further, $R_{I_\alpha}(\lambda, \mu)$ is readily seen to be smooth in α and satisfy $\|\partial_\alpha R_{I_\alpha}\|_{L^\infty(\mathbb{R}^2)} \leq C$ for α bounded. The latter leads to

$$\mathcal{S}_3 \leq C' \cdot \text{Lip}[f] \cdot |\widehat{q} - q| \cdot \frac{N}{L} \xrightarrow{L \rightarrow +\infty} 0. \quad (4.7)$$

Clearly, $\mathcal{S}_2 = O(L^{-1})$ since f is bounded while

$$|\mathcal{S}_1| \leq \frac{N}{L} \cdot \text{Lip}[f] \cdot \|\widehat{\xi}^{-1} - \xi_0^{-1}(\cdot \mid \widehat{q})\|_{L^\infty(\xi_0(I_{\Lambda+\epsilon|q}))} = O(L^{-1}). \quad (4.8)$$

I now focus on the remaining cases, *viz.* $D = 1/2$ and $-1 < \Delta \leq 1$ and picking some $M > 0$ decompose the sum of interest as

$$\frac{1}{L} \sum_{a=1}^N f(\lambda_a) = \frac{1}{L} \sum_{|\lambda_a| > M} f(\lambda_a) + \frac{1}{L} \sum_{a: |\lambda_a| \leq M} \left[f \circ \widehat{\xi}^{-1} \left(\frac{a}{L} \right) - f \circ \xi_0^{-1} \left(\frac{a}{L} \mid +\infty \right) \right] + \frac{1}{L} \sum_{|\lambda_a| \leq M} f \circ \xi_0^{-1} \left(\frac{a}{L} \mid +\infty \right). \quad (4.9)$$

The last term converges as a Riemann sum

$$\frac{1}{L} \sum_{|\lambda_a| \leq M} f \circ \xi_0^{-1} \left(\frac{a}{L} \mid +\infty \right) \rightarrow \int_{-M}^M f(\lambda) \rho_\infty(\lambda) d\lambda. \quad (4.10)$$

The estimates of Theorem 3.5 ensure that, provided ϵ is small enough, one has

$$\limsup_{L \rightarrow +\infty} \left| \frac{1}{L} \sum_{|\lambda_a| \leq M} \left[f \circ \widehat{\xi}^{-1} \left(\frac{a}{L} \right) - f \circ \xi_0^{-1} \left(\frac{a}{L} \mid +\infty \right) \right] \right| \leq C \cdot \text{Lip}[f] \cdot \epsilon. \quad (4.11)$$

Finally,

$$\limsup_{L \rightarrow +\infty} \left| \frac{1}{L} \sum_{|\lambda_a| > M} f(\lambda_a) \right| \leq \|f\|_{L^\infty(\mathbb{R})} \cdot \limsup_{L \rightarrow +\infty} \left(\frac{1}{L} \#\{a \in \llbracket 1; N \rrbracket : |\lambda_a| > M\} \right). \quad (4.12)$$

Thus, upon sending $L \rightarrow +\infty$ and then $\epsilon \rightarrow 0^+$ one gets

$$\frac{1}{L} \sum_{a=1}^L f(\lambda_a) \rightarrow \int_{-M}^M f(\lambda) \rho_\infty(\lambda) d\lambda + O \left(\limsup_{L \rightarrow +\infty} \left(\frac{1}{L} \#\{a \in \llbracket 1; N \rrbracket : |\lambda_a| > M\} \right) \right). \quad (4.13)$$

Finally, it remains to relax $M \rightarrow +\infty$. The remainder term, in virtue of Proposition 2.3, tends to 0. ■

4.2 The conformal structure of the low-lying excitations

The XXZ chain embedded in an external magnetic field h refers to the Hamiltonian $H_{\Delta;h} = H_{\Delta} - S^z \frac{h}{2}$ on \mathfrak{h}_{XXZ} . Its eigenvalues and eigenvectors are readily deduced from those of H_{Δ} owing to $[H_{\Delta}, S^z] = 0$. The eigenvalue of $H_{\Delta;h}$ associated with the eigenvector $\Psi(\{\lambda_a\}_1^N)$ of H_{Δ} that is parametrised by the Bethe roots $\{\lambda_a\}_1^N$ takes the form

$$\mathcal{E}(\{\lambda_a\}_1^N) = \left(J\Delta - \frac{h}{2}\right)L + \sum_{a=1}^N \epsilon(\lambda_a) \quad (4.14)$$

with ϵ as defined in (1.49). The large- L expansion of the above eigenvalue involves the effective dressed energy defined as

$$\epsilon_{\text{eff}}(\lambda \mid Q) = \epsilon(\lambda \mid Q) + \epsilon(Q \mid Q) \left(\varphi(Q, \lambda \mid Q) - \varphi(-Q, \lambda \mid Q) \right). \quad (4.15)$$

Proposition 4.2. *Under the assumptions of Theorem 3.2, for any $0 \leq D < 1/2$ and $\Delta > -1$ it holds*

$$\mathcal{E}(\{\lambda_a\}_1^{N+s}) = L \cdot \mathcal{E}_0(\widehat{q}) + \sum_{k=1}^2 \frac{1}{L^{k-1}} \mathcal{E}_k(\widehat{q} \mid \{\widehat{x}_{p'_a}\}_1^{n+s}; \{\widehat{x}_{h_a}\}_1^n) + \mathcal{O}(L^{-2}) \quad (4.16)$$

where

$$\mathcal{E}_0(\widehat{q}) = J\Delta - \frac{h}{2} + \int_{-\widehat{q}}^{\widehat{q}} \epsilon(\mu) \rho(\mu \mid \widehat{q}) \cdot d\mu \quad (4.17)$$

$$\mathcal{E}_1(\widehat{q} \mid \{\widehat{x}_{p'_a}\}_1^{n+s}; \{\widehat{x}_{h_a}\}_1^n) = \sum_{a=1}^{n+s} \epsilon_{\text{eff}}(\widehat{x}_{p'_a} \mid \widehat{q}) - \sum_{a=1}^n \epsilon_{\text{eff}}(\widehat{x}_{h_a} \mid q) \quad (4.18)$$

with ϵ_{eff} as defined in (4.15) and, finally,

$$\mathcal{E}_2(\widehat{q} \mid \{\widehat{x}_{p'_a}\}_1^{n+s}; \{\widehat{x}_{h_a}\}_1^n) = -\frac{\epsilon'_{\text{eff}}(\widehat{q} \mid \widehat{q})}{12 \xi'_0(\widehat{q} \mid \widehat{q})} + \frac{1}{2} \epsilon'_{\text{eff}}(\widehat{q} \mid \widehat{q}) \xi'_0(\widehat{q} \mid \widehat{q}) \cdot \left\{ (q_+^{(1)})^2 + (q_-^{(1)})^2 \right\} \quad (4.19)$$

with $q_{\pm}^{(1)}$ as defined by (3.90).

Proof—

Since ϵ is holomorphic in a neighbourhood of the real axis, one has the representation

$$\begin{aligned} \mathcal{E}(\{\lambda_a\}_1^N) - \left(J\Delta - \frac{h}{2}\right)L &= L \int_{\widehat{q}_L}^{\widehat{q}_R} \epsilon(\mu) \widehat{\xi}'(\mu) \cdot d\mu + \sum_{a=1}^n [\epsilon(\widehat{x}_{p'_a}) - \epsilon(\widehat{x}_{h_a})] \\ &\quad - \sum_{\epsilon=\pm} \int_{\widehat{\Gamma}_{\epsilon}} \frac{\epsilon'(\widehat{\xi}^{-1}(z))}{\widehat{\xi}'(\widehat{\xi}^{-1}(z))} \ln[1 - e^{2i\pi\epsilon Lz}] \cdot \frac{dz}{2i\pi}. \end{aligned}$$

A straightforward application of Watson's lemma leads to

$$\sum_{\epsilon=\pm} \int_{\widehat{\Gamma}_{\epsilon}} \frac{\epsilon'(\widehat{\xi}^{-1}(z))}{\widehat{\xi}'(\widehat{\xi}^{-1}(z))} \ln[1 - e^{2i\pi\epsilon Lz}] \cdot \frac{dz}{2i\pi} = -\frac{\epsilon'(\widehat{q})}{12L \xi'_0(\widehat{q} \mid \widehat{q})} + \mathcal{O}\left(\frac{1}{L^2}\right). \quad (4.20)$$

In its turn, the one-fold integral admits the expansion

$$\int_{\widehat{q}_L}^{\widehat{q}_R} \epsilon(\mu) \widehat{\xi}'(\mu) \cdot d\mu = \sum_{k=0}^2 \frac{1}{L^k} \int_{\widehat{q}_L}^{\widehat{q}_R} \epsilon(\mu) \xi_k'(\mu) \cdot d\mu + O\left(\frac{1}{L^3}\right) \quad (4.21)$$

where, so as to lighten the notation, I have dropped the dependence of the $\xi_k(\lambda)$ on the roots \widehat{x}_{p_a/h_a} . It then remains to use the Taylor expansion

$$\int_{\widehat{q}_L}^{\widehat{q}_R} f(\mu) \cdot d\mu \simeq \int_{-\widehat{q}}^{\widehat{q}} f(\mu) \cdot d\mu + \sum_{k \geq 0} \left\{ \frac{f^{(k)}(\widehat{q})}{(k+1)!} (\widehat{q}_R - \widehat{q})^{k+1} - \frac{f^{(k)}(-\widehat{q})}{(k+1)!} (\widehat{q}_L + \widehat{q})^{k+1} \right\}, \quad (4.22)$$

and the differential identities satisfied by the dressed phase (1.53) so as to get the claim after some straightforward algebra. \blacksquare

Lemma 4.1. *Let $h_c > h > 0$ for $-1 < \Delta \leq 1$ and $h_c > h \geq h_c^{(L)}$ for $1 < \Delta$, then the function $Q \mapsto \mathcal{E}_0(Q)$ attains a unique minimum at $Q_F \in I_t \cap \mathbb{R}^+$, the so-called Fermi boundary of the model. Furthermore, Q_F is such that*

$$\varepsilon(\lambda | Q_F)_{|I_{Q_F}} < 0 \quad , \quad \varepsilon(\lambda | Q_F)_{|I_{Q_F}^c} > 0 \quad \text{and} \quad \varepsilon(Q_F | Q_F) = 0. \quad (4.23)$$

Proof —

It follows from straightforward algebra that

$$(\mathcal{E}_h^{(0)})'(Q) = \varepsilon(Q | Q) \cdot \rho(Q | Q). \quad (4.24)$$

Since, $\rho(Q | Q) > 0$, the derivative has the sign of $\varepsilon(Q | Q)$. It follows from Proposition 1.2 that $\varepsilon(Q | Q) < 0$ on $[0; Q_F[$ and $\varepsilon(Q | Q) > 0$ on $I_{Q_F}^c$. This entails that $Q \mapsto \mathcal{E}_0(Q)$ admits a unique minimum at Q_F . \blacksquare

Define the Fermi density \bar{D}_F by $D_F = p(Q_F | Q_F)$. Given L large enough it seems reasonable to expect that the ground state of $H_\Delta^{(N)}$ with N such that $|D_F - N/L|$ is minimal will give rise to a † ground state of $H_{\Delta;h}$. In the present state of the art, I am however not in position to prove the statement due to the lack of a sufficient uniform in L, N control close to $N/L = 1/2$ of $\mathcal{E}(\{\lambda_a\}_1^N)$.

Theorem 4.3. *Let h be such that $D_F \in]0; 1/2[$ and assume that N, L are such that $D_F - N/L = O(L^{-3})$. Let s be fixed and consider the solution to the logarithmic Bethe equations $\{\lambda_a\}_1^{N+s}$ subordinate to the choice of particle-hole integers $\{p_a\}_1^n$ and $\{h_a\}_1^n$*

$$p_a = 1 - p_a^- \quad \text{for } a = 1, \dots, n_p^- \quad \text{and} \quad p_{n_p^-+a} = N + s + p_a^+ \quad \text{for } a = 1, \dots, n_p^+, \quad (4.25)$$

with $p_a^\pm \in \mathbb{N}^*$ fixed in N, L ,

$$h_a = h_a^- \quad \text{for } a = 1, \dots, n_h^- \quad \text{and} \quad h_{n_h^-+a} = N + s + 1 - h_a^+ \quad \text{for } a = 1, \dots, n_h^+, \quad (4.26)$$

with $h_a^\pm \in \mathbb{N}^*$ fixed in N, L , and $n_h^+ + n_h^- = n = n_p^+ + n_p^-$.

Then, it holds

$$\begin{aligned} \mathcal{E}(\{\lambda_a\}_1^{N+s}) - L\mathcal{E}_0(Q_F) &= \frac{v_F}{L} \left\{ -\frac{1}{12} + \ell^2 [Z^2(Q_F | Q_F) - 1] + \frac{s^2}{4Z^2(Q_F | Q_F)} \right. \\ &\quad \left. + \sum_{\epsilon=\pm} \left(\sum_{a=1}^{n_p^\epsilon} [p_a^\epsilon - 1/2] + \sum_{a=1}^{n_h^\epsilon} [h_a^\epsilon - 1/2] \right) \right\} + O\left(\frac{1}{L^2}\right) \end{aligned} \quad (4.27)$$

† . There could, in principle, more than one ground state since $|D_F - N/L|$ could attain two minima for some fixed value of L .

where

$$v_F = \frac{\varepsilon'(Q_F | Q_F)}{p'(Q_F | Q_F)} \quad (4.28)$$

refers to the velocity of the excitation lying on Fermi boundary.

Proof —

Observe that the choice of integers given in the statement of the theorem always gives rise to a solution of the logarithmic Bethe equations since the conditions stated in Proposition 2.1 are always verified provided that L is large enough. Further, since $D \mapsto q(D)$ is smooth, it holds that $Q_F - \widehat{q} = O(L^{-3})$ where \widehat{q} is the magnetic Fermi boundary associated with $\widehat{D} = N/L$. Also, since for any fixed $a \in \mathbb{Z}$

$$\lim_{L \rightarrow +\infty} \frac{N+a}{L} = D_F \quad \text{and} \quad \lim_{L \rightarrow +\infty} \frac{a}{L} = 0 \quad (4.29)$$

one has

$$\lim_{L \rightarrow +\infty} \widehat{x}_{N+a} = Q_F \quad \text{and} \quad \lim_{L \rightarrow +\infty} \widehat{x}_a = -Q_F. \quad (4.30)$$

These pieces of information are enough so as to obtain the large- L expansion of the particle-hole roots. One gets

$$\widehat{x}_{N+a} = Q_F + \frac{a - \xi_1^{(s)}(Q_F)}{L\xi_0'(Q_F | Q_F)} + O\left(\frac{1}{L^2}\right) \quad \widehat{x}_a = -Q_F + \frac{a - \xi_1^{(s)}(-Q_F)}{L\xi_0'(-Q_F | Q_F)} + O\left(\frac{1}{L^2}\right) \quad (4.31)$$

Note that, above, the dependence of $\xi_1^{(s)}$ on the position of the particle-hole roots is kept implicit. Inserting the expansion of the roots up to $O(L^{-1})$ one gets that

$$\xi_1^{(s)}(\lambda | \{\widehat{x}_{p_a}\}_1^n; \{\widehat{x}_{h_a}\}_1^n) = \frac{1}{2} - \ell\{Z(\lambda | Q_F) - 1\} + \frac{s}{2}\{1 - \varphi(\lambda, Q_F | Q_F) - \varphi(\lambda, -Q_F | Q_F)\} + O\left(\frac{1}{L}\right). \quad (4.32)$$

Straightforward algebra based on these expansions leads to the claim. ■

Conclusion

This paper develops tools allowing one to prove the existence and form of the large- L asymptotic expansion of the counting function associated with the XXZ spin-1/2 chain. The method is robust in that it does not rely on details of the model such as the sign or the magnitude of the L^∞ norm of the derivative of the bare phase. As such it allows one to step out of the setting where the model's Yang-Yang action is strictly convex or where the model is a small perturbation of a non-interacting model. For these reasons it appears plausible that the method of large- L analysis proposed in this paper is applicable to many other quantum integrable models, this independently of the value of their coupling constant, *viz.* the sign of the Lieb kernels.

The method should also allow one to study the case of excited states above the ground state described by Bethe roots such that a fixed number thereof is complex valued. I plan to investigate this issue in a forthcoming publication.

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A Auxiliary results

Below, by f holomorphic on F with F not open, it is meant that f is a holomorphic function on some open neighbourhood of F . Also, given a segment I the box-open neighbourhood $\mathcal{S}_{\eta,\epsilon}(I)$ is as defined in (3.10) while $\partial\mathcal{S}_{\eta,\epsilon}(I)$ stands for the canonically oriented boundary of $\mathcal{S}_{\eta,\epsilon}(I)$.

Proposition A.1. *Let $I = [a; b]$ be a segment and X be a compact in \mathbb{C} such that $I \subset \text{Int}(X)$. Let f_L be a sequence of holomorphic functions on X converging to a holomorphic function f on X . Assume that f is such that $\text{sgn}(f'_{|[a;b]})$ is constant. Then, there exists $\eta, \epsilon > 0$ such that*

$$f : \mathcal{S}_{2\eta,2\epsilon}(I) \rightarrow f(\mathcal{S}_{2\eta,2\epsilon}(I)) \text{ is a biholomorphism.} \quad (\text{A.1})$$

Furthermore, there exists L_0 such that for any $L \geq L_0$ one has the inclusions

$$f_L(I) \subset f(\mathcal{S}_{\eta,\epsilon}(I)) \subset f_L(\mathcal{S}_{2\eta,2\epsilon}(I)) \quad (\text{A.2})$$

and

$$f_L : \mathcal{S}_{2\eta,2\epsilon}(I) \cap f_L^{-1}(f(\mathcal{S}_{\eta,\epsilon}(I))) \rightarrow f(\mathcal{S}_{\eta,\epsilon}(I)) \quad (\text{A.3})$$

is a biholomorphism. Also, there exists a constant $u_{\eta,\epsilon}[f]$ depending on f such that, for any $L \geq L_0$, it holds

$$\inf_{\lambda \in \partial\mathcal{S}_{2\eta,2\epsilon}(I)} \inf_{z \in f(\mathcal{S}_{\eta,\epsilon}(I))} |f_L(\lambda) - z| \geq u_{\eta,\epsilon}[f] > 0. \quad (\text{A.4})$$

Finally, for any $z \in f(\mathcal{S}_{\eta,\epsilon}(I))$, it holds

$$f_L^{-1}(z) = \oint_{\partial\mathcal{S}_{2\eta,2\epsilon}(I)} \frac{\lambda \cdot f'_L(\lambda)}{f_L(\lambda) - z} \cdot \frac{d\lambda}{2i\pi} \quad \text{and} \quad \|f_L^{-1} - f^{-1}\|_{L^\infty(f(\mathcal{S}_{\eta,\epsilon}(I)))} \leq C[f] \cdot \|f_L - f\|_{L^\infty(X)} \quad (\text{A.5})$$

for some constant $C[f] > 0$ only depending on f, ϵ, η and the segment I .

Proof —

I first establish the statement relative to f . It is enough to consider the case where $f'_{|[a;b]} > 0$. Then, by continuity, there exists $\tilde{\eta}, \epsilon > 0$ such that $\mathcal{S}_{3\tilde{\eta},3\epsilon}(I) \subset X$ and $f'(\lambda) \neq 0$ for any $\lambda \in \overline{\mathcal{S}_{3\tilde{\eta},3\epsilon}(I)}$. Hence, f is a local biholomorphism on $\mathcal{S}_{3\tilde{\eta},3\epsilon}(I)$. Thus it remains to show that there exists $\eta, \tilde{\eta} > \eta > 0$ such that $f|_{\mathcal{S}_{3\eta,3\epsilon}(I)}$ is injective. If not, then there exists two sequences

$$z_1^{(n)} \neq z_2^{(n)}, \quad a - 3\epsilon < \Re(z_k^{(n)}) < b + 3\epsilon \quad \text{such that} \quad \Im(z_k^{(n)}) \xrightarrow{n \rightarrow +\infty} 0 \quad \text{and} \quad f(z_1^{(n)}) = f(z_2^{(n)}). \quad (\text{A.6})$$

By compactness of $\overline{\mathcal{S}_{2\tilde{\eta},2\epsilon}(I)}$ one may build converging subsequences $z_a^{(p_n)} \rightarrow x_a \in [a - 3\epsilon; b + 3\epsilon]$. Yet since $f'_{|[a-3\epsilon;b+3\epsilon]} > 0$, f is bijective on $[a - 3\epsilon; b + 3\epsilon]$ and thus $f(x_1) = f(x_2)$ implies that one has $x_1 = x_2 = x$. There exists an open neighbourhood U_x of x such that $f|_{U_x}$ is a biholomorphism. For n large enough, one has that $z_a^{(p_n)} \in U_x$, $a = 1, 2$. Thence, by injectivity of f on U_x , $z_1^{(p_n)} = z_2^{(p_n)}$ for n large enough, a contradiction. Thus f is a biholomorphism on $\mathcal{S}_{3\eta,3\epsilon}(I)$ for some $\eta > 0$ and, *a fortiori* (A.1) is established.

It remains to establish the statements relative to f_L . Since f is injective on $\mathcal{S}_{3\eta,3\epsilon}(I)$, it follows that $f(\partial\mathcal{S}_{2\eta,2\epsilon}(I)) \cap \overline{f(\mathcal{S}_{\eta,\epsilon}(I))} = \emptyset$. Thus, by compactness of these sets, one has

$$2u_{\eta,\epsilon}[f] = \inf_{z \in \mathcal{S}_{\eta,\epsilon}(I)} \inf_{s \in \partial\mathcal{S}_{2\eta,2\epsilon}(I)} |f(z) - f(s)| > 0. \quad (\text{A.7})$$

Furthermore, the invertibility of f on $\mathcal{S}_{2\eta,2\epsilon}(I)$ also ensures that

$$2\iota_{\eta,\epsilon}[f] = \inf \left\{ |f'(\lambda)| : \lambda \in \mathcal{S}_{2\eta,2\epsilon}(I) \right\} > 0. \quad (\text{A.8})$$

Since $f_L \rightarrow f$ on X , one has that there exists L_0 such that

$$\|f_L - f\|_{L^\infty(\partial\mathcal{S}_{3\eta,3\epsilon}(I))} < \min\{u_{\eta,\epsilon}[\xi]/C_1, \iota_{\epsilon,\eta}[f]\} \quad \text{provided that } L \geq L_0. \quad (\text{A.9})$$

Here, C_1 is such that

$$\|f'\|_{L^\infty(\mathcal{S}_{2\eta,2\epsilon}(I))} \leq C_1 \cdot \|f'\|_{L^\infty(\partial\mathcal{S}_{3\eta,3\epsilon}(I))}. \quad (\text{A.10})$$

This bound implies (A.4) and ensures that $2\iota_{\eta,\epsilon}[f_L] > \iota_{\eta,\epsilon}[f] > 0$ so that f_L is a local biholomorphism on $\mathcal{S}_{2\eta,2\epsilon}(I)$. Furthermore, in virtue of Rouché's theorem,

$$\text{for any } z \in f(\mathcal{S}_{\eta,\epsilon}(I)) \quad s \mapsto f_L(s) - z \quad (\text{A.11})$$

has exactly one simple zero in $\mathcal{S}_{2\eta,2\epsilon}(I)$, and is thus injective on this set. Therefore, it follows that

$$f_L : \mathcal{S}_{2\eta,2\epsilon}(I) \cap f_L^{-1}(f(\mathcal{S}_{\eta,\epsilon}(I))) \rightarrow f(\mathcal{S}_{\eta,\epsilon}(I)) \quad (\text{A.12})$$

is a biholomorphism. The above also implies that $f(\mathcal{S}_{\eta,\epsilon}(I)) \subset f_L(\mathcal{S}_{2\eta,2\epsilon}(I))$. Also, since f is strictly increasing on $[a - \epsilon; b + \epsilon]$ it is clear that for L large enough $f([a - \epsilon; b + \epsilon]) \supset f_L(I)$. From there the statement about inclusion follows.

Finally, the bounds (A.7) and (A.9) ensure that the integral representation (A.5) for f_L holds. The same representation holds with $f \leftrightarrow f_L$ so that

$$f_L^{-1}(z) - f^{-1}(z) = \oint_{\partial\mathcal{S}_{2\eta,2\epsilon}(I)} \lambda \cdot \frac{f'_L(\lambda)(f(\lambda) - z) - f'(\lambda)(f_L(\lambda) - z)}{(f_L(\lambda) - z)(f(\lambda) - z)} \cdot \frac{d\lambda}{2i\pi}. \quad (\text{A.13})$$

The difference is then bounded thanks to (A.4), (A.7), (A.9) and the bound (A.10). ■

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